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Quantum Fields and Fundamental Forces MSc

SELF-COUPLING OF SPIN-2: EINSTEIN'S, UNIMODULAR AND MASSIVE THEORIES OF GRAVITY

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Abstract

This dissertation considers theories of spin-2 fields propagating in flat spacetime with *Diff* and *TDiff* gauge symmetry [3]. Non-linear action terms associated with self-interactions are explored. These are obtained by coupling the fields with their own energy-momentum, i.e. the free field equation of motion is sourced by the energy-momentum tensor. It is shown that, depending on the starting point, this gives rise to theories of gravity invariant under general coordinate transformations, namely GR, and unimodular coord. transf.. The question whether this principle comprises a non-geometric derivation of Einstein's theory, in particular whether this leads uniquely to GR, is considered. It is also shown that including a mass term and a kinetic term with *Diff* symmetry, dRGT massive gravity [4, 5] (with the set of free parameters partially fixed) can be generated.

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Through the darkness of future past,

the magician longs to see ...

Contents

1 Introduction

If one were to open a popular textbook on Einstein's Theory of General Relativity (GR) [1] one would be likely to find claims such as the following (from [6], p.436):

> *Just as one can "descend" from GR to linearized theory by linearizing about flat spacetime (...), so one can "bootstrap" one's way back up from linearized theory to general relativity by imposing consistency between the linearized field equations and the equations of motion, or, equivalently, by asking about: (1) the stress-energy carried by the linearized gravitational field $h_{\mu\nu}$; (2) the influence of this stress-energy acting as a source for corrections $h^{(1)}_{\mu\nu}$ to the field; (3) the stress-energy carried by the corrections $h^{(1)}_{\mu\nu}$; (4) the influence of this stress-energy acting as a source for corrections $h^{(2)}_{\mu\nu}$ to the corrections $h^{(1)}_{\mu\nu}$; (5) the stress-energy carried by the corrections to the corrections; and so on.*

The reference book by Misner, Thorne and Wheeler brings this up as an "alternative way to derive general relativity (from spin-2 viewpoint)" aside from "Einstein derivation (from geometric viewpoint)". As is well-known the lowest-order term in a (functional[1]) Taylor expansion of the Einstein-Hilbert action $S_{\mathrm{EH}}[g]$ around the flat Minkowski metric ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$), or equivalently the linear approximation to Einstein's field equation, coincides with Fierz-Pauli theory of massless spin-2 in flat spacetime [7]. Also in [6] (p. 424) there's a summary of Deser's 1970 work [8], one of the most relevant papers concerning the derivation of GR by a process like the one described in the previous paragraph.

These claims on textbooks are part of a general belief

(i) on the possibility of obtaining full (non-linear) GR by coupling a Fierz-Pauli (FP) field/graviton (consistently, as in the quotation above) to the total energy-momentum tensor, while only using "standard concepts in Lorentz invariant field theory" and withholding "any geometrical assumptions";

(ii) and on this process leading uniquely to Einstein's theory

as stated in [9], where it is argued that the first point is false (*it is not possible to obtain the Einstein-Hilbert action starting from the standard graviton action and iterating ...*) and hence the second too. In [10], this is referenced as "Padmanabhan's 2008 thought-provoking analysis" which "raised some concerns that are having resonance in the community", including a reply by Deser [11] where he stands up for his *non-geometric derivation* [8] *of GR (...) as the unique consistent self-interacting system, extending the initial free massless spin-2.* The piece

[1] In the rest of the text we'll refer to this simply by "Taylor expansion".

of conventional wisdom regarding this topic that is undoubtedly wrong is it being already well understood.

This iterative self-coupling process has been called *Gupta Program* after the 1954 paper [12]. Along with Gupta, behind some of the first attempts at deriving GR non-geometrically was Robert Kraichnan who worked on the topic for his 1946-47 Bachelor's thesis but only in 1955 published [13]. Presumably unaware of Gupta's and Kraichnan's work, Feynman also contributed via his 1962-63 Lectures on Gravitation [14]. One of the few times Deser agrees with Padmanabhan in [11] concerns the assumptions the early derivations feed on since until [8] none performed the iterations all the way (*they were replaced by such statements as "what else could it sum to?" and "the sum must be general covariant, ergo GR"*, quoting Deser). Some authors decided to follow a distinct path towards GR starting with FP spin-2 theory; like Wald [15] and Fang [16], who used the fact that diffeomorphism invariance is the only nonlinear deformation of the gauge symmetry in the linear theory ([17]) and Weinberg [18], who focused on graviton-graviton scattering amplitudes.

The present work is minimalist when it comes to analise, draw comparisons or criticise non-geometric derivations of GR like the aforementioned ones. We choose to do it briefly throughout the body of this dissertation but some remarks are left to be discussed in the conclusion. Nevertheless we would like to make a comment on [9] that, to the best of our knowledge, has not been made in published work. In p. 11, in order to obtain the "Γ^2 action" (that differs from the Einstein-Hilbert action by a surface term), Padmanabhan states: *We merely use the fact that the analysis leading to Eq. (41) was completely independent of the form of* A_0 *as well as the nature of the fields* ϕ_A; however δA_1, δA_2, etc, was integrated in p. 10 under the assumption that ϕ_A is not h_{ab} (or, equivalently, A_0 doesn't depend on h_{ab}). Hence we disregard Padmanabhan's remarks involving the object $\mathcal{S}^{ab}$.

The GR bootstrap stands as the motivation for this work and its backbone. Nonetheless, we incorporate massive theories and actions invariant under Unimodular Coordinate Transformations (UCTs)[2], in opposition to S_{EH} which is invariant under General Coordinate Transformations (GCTs) - hence the name "Unimodular Gravity". Many authors une use this to refer to theories like the one presented by Einstein [2] (three years after GR) where the determinant of the metric either is non-dinamical from the start or it can be gauge-fixed.

- We start in section 2.1 by exploring two routes (that we call "covariantization" and "U-covariantization") towards field theories in non-inertial coordinates that are dinamically equivalent to a certain special-relativistic field theory (SRFT);

[2]which correspond to scalar-tensor theories.

- Section 2.2 is inspired by [3], where *the most general Lorentz invariant Lagrangian for a massless graviton* is considered and it's shown that, to avoid ghosts and classical instabilities, TDiff[3] gauge invariance is necessary (but not sufficient). We then start with the entire TDiff invariant family of kinetic action terms for a field h^{ab} of the FP type and derive the associated Gauge/Bianchi Identities. Even though there are two possible enhancements of this gauge symmetry, we only focus on the one that we already know from FP theory. In section 2.3, we make our h^{ab} field interact with other fields and examine how they can couple;

- Chapter 3 is entirely dedicated to the object known as the Energy-Momentum Tensor (EMT) and modifications of it, while exploring the connection between these and Gauge/Bianchi Identities;

- In section 4.1 we start by completing section 2.3 and then in 4.2 we apply in concrete terms the iterative procedure associated with self-coupling the FP h^{ab} field (the rest of FP-type theories, i.e. the ones with TDiff gauge invariance, are left for later);

- We have been exalting the bootstrap side of deriving GR non-geometrically but note that this implies an approach to GR much closer to a particle physicist view of Einstein's theory (see the nice introduction in [24] for some historical context). In section 4.3, we introduce a formulation of GR that employs two connections (as presented in [25]) such that we call it "Bi-connection GR". This serves to convince the reader that what we have obtained in the previous section is indeed the Einstein-Hilbert action;

- In section 4.4, we perform a reverse engineering from a metric theory of gravity to find out how the gauge symmetry in the linear FP theory and its coupling to the EMT arise. This is replicated in section 5.1 starting with a variation of the Einstein-Hilbert action where the lagrangian is multiplied by a smooth function of the determinant of the metric, entailing invariance under UCTs;

- The iterative procedure for TDiff theories of propagating spin-2 gravitons (and a scalar) is carried out in section 5.2. Since we ended up discussing mass terms in theories with TDiff invariant kinetic terms and self-coupling the massive spin-2 field accordingly, we decided to address FP theory with arbitrary mass terms, in section 5.3;

- The approach we took in the previous section, failed to avoid ghosts. In order to over-

[3]The author views *transverse Fierz-Pauli* as a much better designation when compared with the term behind TDiff: transverse diffeomorphism (which is more accurately employed in the context of page 8); however the diminuitives TDiff and Diff are useful and are already spread in the literature.

come this, we try out representing the background spacetime by a *tetrad/vielbein* (from section 2.1 point of view, this corresponds to a "partial" covariantization). In this framework we expect that self-coupling would lead to a theory of, what we call, "tetrad gravity" equivalent to GR. Sections 6.1, 6.2 and 6.3 are analogous to 3.2, 4.4 and 5.3, respectively.

We intend the body of the dissertation, that we've just summarised, to be as self-contained as possible. Even though at the end of it the reader may be convinced of (ii) from p. 1, we advise to wait for the conclusion.

There are some papers co-authored by Deser that expand his 1970 work and may be of interest for the reader - [26–29] - and there is also [30], more recent, where like in the present work TDiff is considered (though their approach to self-coupling differs from ours). **Conventions** for the epsilon symbol $\tilde{\epsilon}$ and covariant derivatives are those of textbook [31]. Symmetrization and antisymmetrization are performed with unit weight.[4]

[4]Also, in what follows, we tend to use italic and quotation marks for terms coined by others and by us, respectively.

2 Theories with FP-type field

2.1 Special-relativistic field theories

The quantity of action (traditionally denoted by the letter "S", following a paper by Hamilton [19]) is one of the most important elements in modern theoretical Physics. This text makes no exception (on the contrary) as we will be continuously dealing with field theories. By action we mean a functional of classical fields obtained by integrating a local quantity depending on the fields and its derivatives - the lagrangian (density) - over spacetime (manifold). The dynamics of a classical field theory are given by a set of *equations of motion* (EOMs) determined by requiring stationarity[5] of the action under arbitrary variations of each so-called "dynamical" field.

Consider a SRFT given by

$$\mathscr{A}[\eta; h] = \int d^D x \, \mathscr{L}[\eta^{ab}; h^{ab}, \partial_c h^{ab}] \tag{2.1}$$

(to distinguish the actions associated with SRFTs we'll use not "S" but $\mathscr{A}$). Its dynamical fields $h^{ab}(x)$ live on a flat spacetime of dimension D with metric

$$\eta_{ab}(x) = Diag(-1, 1, 1, 1, ...) \tag{2.2}$$

but we will always use the inverse metric η^{ab} as our fundamental variable. One should not assume $\mathscr{A}$ to be invariant under General Coordinate Transformations (GCTs), that's why when we introduced h^{ab} we indicated a particular set of coordinates[6]. The informations that $x \equiv x^a$ is special is conveyed in the fact that η is always assumed to be given by (2.2) such that this set is called *inertial coordinates*.

When indicating functional dependence, we'll not write the indices on the components of the fields. On the other hand, lagrangians' variables will always hold them explicit. In plain text, we'll often ommit indices.

We restrict ourselves to SRFTs where h^{ab} ($= h^{ba}$) is a tensorial representation of $SO(1, D-1)$, the group of (global) Lorentz Transformations. The action $\mathscr{A}$ is invariant under the coordinate transformations belonging to the Poincaré group (Lorentz Group + translations).

[5]The boundary conditions for the "stationary action principle" set these variations to vanish on the boundary (or at infinity).

[6]We'll keep doing this throughout the text but for a different reason. Later we'll be dealing with tensors such that to write its components we must have chosen an atlas.

These leave η^{ab} invariant (they are the isometries of Minkowski space) such that our special-relativistic metric is fixed. In this section, we focus on how we can generalize the action and its variables from inertial to curvilinear coordinates.

Under Poincaré transformations, a Lorentz tensor with two upper indices transforms in the following way:

$$h^{ab}(x) \to h'^{ab}(x') = \Lambda^a{}_c \Lambda^b{}_d h^{cd}(x) \tag{2.3}$$

where

$$x'^a = \Lambda^a{}_b x^b - t^a \quad \text{with} \quad \Lambda \in SO(1,3) \;\; (det(\Lambda^a{}_b) \equiv |\Lambda| = 1) \tag{2.4}$$

and $\Lambda^a{}_b$ and t^a are constants over spacetime. Assuming the transformations are infinitesimal:

$$\tilde{\delta} h^{ab} \equiv h'^{ab}(x') - h^{ab}(x) = 2h^{c(a}(x)\epsilon^{b)}{}_c + O(\epsilon^2) \tag{2.5}$$

since $\Lambda^a{}_c = \delta^a_c + \epsilon^a{}_c$ with $\epsilon^a{}_c = -\epsilon_c{}^a$. Writing

$$\Lambda^a{}_c x^c = x^a + \epsilon^a{}_c x^c \equiv x^a - \varepsilon'^a(x) \tag{2.6}$$

we see that $\epsilon^a{}_b = -\partial_b \varepsilon'^a(x)$. Also, writing

$$\begin{aligned} x^a = x'^a + \varepsilon'^a(x) + t^a \equiv& x'^a + \varepsilon^a(x) \\ =& x'^a + \varepsilon^a(x') + O(\varepsilon^2) \,, \end{aligned} \tag{2.7}$$

we have

$$\delta h^{ab}(x') = \varepsilon^c(x')\partial_c h^{ab}(x') - 2h^{c(a}(x')\partial_c \varepsilon^{b)}(x') + O(\varepsilon^2) \tag{2.8}$$

where $\delta h^{ab}(x) \equiv h'^{ab}(x) - h^{ab}(x)$. To account for the fact that the Lorentz and translation parameters $\epsilon^a{}_b$ and t^a, respectively, are infinitesimal is enough for ε^a to also be it. After discarting quadratic and higher order terms, we have

$$\delta h^{ab} = \varepsilon^c \partial_c h^{ab} - 2h^{c(a}\partial_c \varepsilon^{b)} \tag{2.9}$$

where the parameter ε is uniquely defined by

$$\begin{aligned} &a) \quad \Box \varepsilon^a = 0 \\ &b) \quad \partial_a \varepsilon^a = 0 \end{aligned} \tag{2.10}$$

Writing ε as a polynomial in x, $a)$ forces it to be of linear order at most and then $b)$ leads, according to (2.6), to the antisymmetry of the coefficient of the linear order term. Looking at condition $b)$ independently, it forces the transformation to be "unimodular":

$$\left|\frac{\partial x'^a}{\partial x^b}\right| = \left|\frac{\partial(x^a - \varepsilon^a)}{\partial x^b}\right| = |\delta^a_b - \partial_b \varepsilon^a| = 1 - \delta^b_a\,(\partial_b \varepsilon^a) = 1 \tag{2.11}$$

From (2.9), one sees that the Poincaré group is a subgroup of the group of GCTs[7] where the parameter $\varepsilon \equiv \xi$ is not constrained by (2.10). Now, we can fulfil our goal of having a representation of a group of coordinate transformations that take us to non-inertial coordinates. We have only to choose, for each object in the action, representations of the GCTs group for which our previous representation of the Poincaré group is a subrepresentation. This way, we're converting "Lorentz tensors" into "spacetime tensors" (the usual tensors, i.e. w.r.t GCTs).

The relaxing of condition a) makes the transformations local. Hence if the action in arbitrary coordinates is invariant under GCTs, choosing inertial coordinates will just ammount to fix a gauge such that we retrieve our SRFT. However, since the Minkovski metric obeys the flatness requirement $\mathcal{R}^a{}_{bcd} = 0$ (valid for any coordinate system), choosing inertial coordinates by performing a GCT will only be possible if $\bar{g}$ (the spacetime tensor obtained from the Lorentz tensor η according to the previous paragraph) is already flat.

For the action to be invariant under GCTs, all "spacetime indices" (that before were "Lorentz's") must be contracted. However this is not enough as there's an essential part of our SRFT's action - the volume element $d^D x$ - that while transforming as a scalar under Lorentz Transformations, under GCTs it transforms in a different way from tensors:

$$\begin{aligned} d^D x \to d^D x' &= d^D x \left| \frac{\partial x'^a}{\partial x^b} \right| \\ &= d^D x \, (1 - \partial_a \xi^a) \Rightarrow \delta d^D x = -d^D x \, \partial_a \xi^a \end{aligned} \tag{2.13}$$

The volume element transforms like what is called a scalar density of weight $\omega = -1$. Let's say that it belongs to the weight-ω (n,m)-tensor density representation of GCTs[8] where $n = 0 = m$ and $\omega = -1$, such that our spacetime tensors are weight-0 tensor densities. For the action to be invariant under this representation, the lagrangian must be a weight-1 scalar density. By assigning a weight to each action variable, this can be achieved in numerous ways and

[7]This was actually obvious from the fact that Poincaré transformations are coordinate transformations: $h'^{ab}(x') = \Lambda^a{}_c \Lambda^b{}_d h^{cd}(x)$ can be written as

$$h'^{ab}(x') = \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} h^{cd}(x) \tag{2.12}$$

with x' as given by (2.4).

[8]schematically (with TD standing for tensor density):

$$\begin{aligned} \{\text{weight-}\omega \ (n,m)\text{-TD rep.}\} &= \{\text{weight-0 } (n,m)\text{-TD rep.}\} \otimes \{\text{weight-}\omega \ (0,0)\text{-TD rep.}\} \\ &\equiv \{(n,m)\text{-“tensor rep.”}\} \otimes \{\text{weight-}\omega \text{ “density rep.”}\} \end{aligned}$$

this time we don't have any orientation since our SRFT is in principle only invariant under density subrepresentations (c.f. footnote 4) which are trivial (and "all representations have a trivial subrepresentation"). However, any weight-ω (n,m)-tensor density can be written as the product of a (n,m)-tensor and the $\omega/2$ power of the metric's determinant (modulus). We then choose to convert our volume element into a weight-0 scalar

$$d^D x \to d^D x \sqrt{-|\bar{g}|} \tag{2.14}$$

such that $d^D x$ is recovered when $\bar{g} = \eta$ and mantain every action variable as a weight-0 tensor. Such a action is invariant under GCTs. (Something similar to what we've done here is employed when finding a representation of the GCTs group for action variables that involve a derivative. For example, $\partial_c h^{ab}$ is not a spacetime tensor so we build a connection using the metric and its derivatives - the Levi-Civita connection[9] - and use it to obtain a covariant derivative such that $\bar{\nabla}_c h^{ab}$ is a spacetime tensor. One recovers $\partial_c h^{ab}$ when $\bar{g} = \eta$). The theory was "covariantized"! (In this text, we generally say something is covariant when it transforms in a representation of the group of GCTs).

We now address the fact that the Poincaré group is also a subgroup of the group of UCTs, also know as *Transverse Diffeomorphisms*, such that $\varepsilon = \boldsymbol{\xi}$ is constrained by $b)$ in (2.10). Representations of this group also take us to non-inertial coordinates:

$$\begin{aligned} &\eta'^{ab} = \eta^{ab} + \boldsymbol{\xi}^c \partial_c \eta^{ab} - 2\eta^{c(a} \partial_c \boldsymbol{\xi}^{b)} \\ &\Rightarrow \delta|\eta| = -|\eta| \eta_{ab} \delta\eta^{ab} = -|\eta| \boldsymbol{\xi}^c \partial_c D + 2|\eta| \partial_c \boldsymbol{\xi}^c = 0 \end{aligned} \tag{2.15}$$

where $\partial^a \equiv \eta^{ac}\partial_c$. The fact that the any metric's determinant is invariant under UCTs could also be seen from the fact that this transforms under a GCT like a scalar density but the only density representation that is also a representation of UCTs is the trivial one, according to (2.11) (under UCTs, a (n,m)-tensor density transforms like a (n,m)-tensor and we call it an U-tensor). Like the flatness requirement for our metric in non-inertial coordinates $\bar{\mathfrak{g}}$, that sill holds here, we must impose that $|\bar{\mathfrak{g}}| = |\eta| = -1$ if we want to retrieve our SRFT by performing a UCT that take us to inertial coordinates (we write $\bar{\mathfrak{g}}$ instead of $\bar{g}$ to indicate that we're using *unimodular coordinates* [32]).

Hence, employing representations of this group, we cannot have an action in completely arbitrary coordinates but in unimodular (otherwise arbitrary) coordinates which is enough for our purposes. This time, to have a generalization of our SRFT action that is invariant under UCTs, is enough for all "spacetime indices" (that before were "Lorentz's") to be contracted.

[9] Ambiguities were faced by choosing to require it to be torsion free and metric compatible.

In converting derivative indices, the covariant derivative $\bar{\boldsymbol{\nabla}}$ must still be employed with a Levi-Civita connection $\boldsymbol{\Gamma}[\bar{\mathfrak{g}}]$. In this case, we say that the theory was "U-covariantized" (when we use the term SRFT we're implying inertial coordinates are employed, otherwise we'll refer to as a covariantized or U-covariantized SRFT). Finally, note that

$$\boldsymbol{\Gamma}^b_{ba} = \frac{\partial_a\sqrt{-|\bar{\mathfrak{g}}|}}{\sqrt{-|\bar{\mathfrak{g}}|}} = 0 \tag{2.16}$$

and this implies $\bar{\boldsymbol{\nabla}}_a\boldsymbol{\xi}^a = \partial_a\boldsymbol{\xi}^a + \boldsymbol{\Gamma}^a_{ab}\boldsymbol{\xi}^b = \partial_a\boldsymbol{\xi}^a = 0$.

2.2 TDiff and Diff gauge symmetries

Consider a specific expression for (2.1):

$$\mathscr{A} = \int d^Dx \left[\frac{-1}{4}\partial_a h_{bc}\partial^a h^{bc} + \frac{1}{2}\partial_a h_{bc}\partial^b h^{ac} + a\frac{-1}{2}\partial_a h^{ab}\partial_b h + b\frac{1}{4}\partial_a h\partial^a h\right] \tag{2.17}$$

where a and b are a couple of arbitrary real parameters. There is another possible way to contract the derivatives of h: $\partial^b h_{bc}\partial_a h^{ac}$; but this would only differ from $\partial_a h_{bc}\partial^b h^{ac}$ by a total derivative - keep in mind actions are defined up to terms that don't contribute to the EOMs such as surface terms (ST). The action $\mathscr{A}$ is gauge invariant under transformation

$$h^{ab} \to h^{ab} + \delta h^{ab} = h^{ab} - 2\partial^{(a}\hat{\xi}^{b)} \tag{2.18}$$

with the parameter constrained by

$$\partial_a\hat{\xi}^a = 0 \text{ [10]} \tag{2.19}$$

This, often refered in the literature as TDiff symmetry, means that the variation of the action under this gauge transformation,

$$\delta\mathscr{A} = \int d^Dx \, \frac{\delta\mathscr{A}}{\delta h^{ab}}\delta h^{ab} + ST \; , \tag{2.20}$$

is null up to ST. Substituting δh^{ab} according to (2.18) and integrating by parts, one obtains

$$\delta\mathscr{A} = 2\int d^Dx \, \hat{\xi}^b\partial^a\frac{\delta\mathscr{A}}{\delta h^{ab}} + ST \tag{2.21}$$

The constrained gauge parameter can be replaced by an arbitrary parameter $\mathcal{F}^{db}$: $\hat{\xi}^b = \partial_d\mathcal{F}^{[db]}$. Then, after integrating again by parts, one has

$$\delta\mathscr{A} = -2\int d^Dx \, \mathcal{F}^{[db]}\partial_d\partial^a\frac{\delta\mathscr{A}}{\delta h^{ab}} + ST \tag{2.22}$$

[10] $\Rightarrow \eta_{ab}\delta h^{ab} = 0$

Taking into account gauge invariance of the action, we then make all surface terms vanish by using appropriate boundary conditions for $\mathcal{F}^{dj}$. Taking into account its arbitrariness, we have the Gauge/Bianchi (off-shell) Identity

$$\partial_{[d}\partial^{a}\frac{\delta\mathscr{A}}{\delta h^{b]a}}=0\Rightarrow\partial^{a}\frac{\delta\mathscr{A}}{\delta h^{ab}}=\partial_{b}\rho \tag{2.23}$$

where ρ is undetermined and corresponds to the scalar degree of freedom (DOF) that was mentioned in the introduction. The discussion we had in section 2.1 suggests that a SRFT is dynamically equivalent to its covariantization or U-covariantization. Hence, to mantain the number of propagating (/dynamical) DOFs, there must be a generalization of transformation (2.18) such that the covariantization or U-covariantization of action (2.17) is gauge invariant under it. Instead of assuming this right ahead, let us start by writing an action S obtained from $\mathscr{A}$ by substituting η by a general metric g, converting Lorentz's into spacetime indices and including a volume element of weight $(\omega-1)$ - $d^{D}x\left(\sqrt{-|g|}\right)^{\omega}$. We then have

$$\delta h^{ab}=-2\nabla^{(a}\check{\xi}^{b)} \tag{2.24}$$

where $\nabla^{a}\equiv g^{ac}\nabla_{c}$ and the gauge parameter is constrained by $\nabla_{a}\check{\xi}^{a}=0$. Two particular aspects of the coordinate derivative in (2.1) and (2.18) were used for $\delta\mathscr{A}$ to vanish (up to ST), namely commutativity and its role in converting volume into surface integrals through Stokes' Theorem:

$$\int_{U}d^{D}x\,\partial_{a}(...)^{a}=\int_{\partial U}d^{D-1}x\,...\equiv ST \tag{2.25}$$

Therefore, for the action in a general background to be gauge invariant under (2.24), the covariant derivative ∇ must, in principle, share these properties. Commutativity is obtained by requiring the background to be flat ($g\to\bar{g}$, $\nabla\to\bar{\nabla}$). For the second one, we must have

$$\int d^{D}x\,\bar{\nabla}_{a}\left[\left(\sqrt{-|\bar{g}|}\right)^{\omega}...\right]^{a}=\int d^{D}x\,\partial_{a}\left[\left(\sqrt{-|\bar{g}|}\right)^{\omega}...\right]^{a} \tag{2.26}$$

Since

$$\bar{\nabla}_{a}\left[\left(\sqrt{-|\bar{g}|}\right)^{\omega}...\right]^{a}=\partial_{a}\left[\left(\sqrt{-|\bar{g}|}\right)^{\omega}...\right]^{a}+\Gamma^{a}_{ab}\left[\left(\sqrt{-|\bar{g}|}\right)^{\omega}...\right]^{b}-\omega\Gamma^{b}_{ba}\left[\left(\sqrt{-|\bar{g}|}\right)^{\omega}...\right]^{a} \tag{2.27}$$

we must set either

$$\begin{aligned} &i)\quad \omega=1 \qquad \text{or}\\ &ii)\quad \Gamma^{b}_{ba}\equiv\mathbf{\Gamma}^{b}_{ba}=0\end{aligned} \tag{2.28}$$

Note that in case ii), $|\bar{g}|\equiv|\bar{\mathfrak{g}}|$ is a constant and the constraint reduces to $\nabla_{a}\check{\xi}^{a}\equiv\bar{\boldsymbol{\nabla}}_{a}\check{\xi}^{a}=\partial_{a}\check{\xi}^{a}=0$. In any case, following (2.20) to (2.23) with the appropriate changes, we end up with the Gauge/Bianchi Identity

$$0 = \bar{\nabla}_{[d}\bar{\nabla}^a \frac{\delta S}{\delta h^{b]a}} = \partial_{[d}\bar{\nabla}^a \frac{\delta S}{\delta h^{b]a}} \tag{2.29}$$

Hence, according to (2.28), we were led straight into covariantization/U-covariantization as was expected. By setting the parameters a and b to 1, we have gauge invariance under (2.18) without any constraint on the parameter (sometimes called Diff symmetry). Hence, to derive the Gauge/Bianchi Identity, $\hat{\xi}$ is already arbitrary and directly from (2.21) we obtain

$$\partial^a \frac{\delta \mathscr{A}}{\delta h^{ab}} = 0 \tag{2.30}$$

instead of (2.23) (implying that $\partial_b \rho = 0$). We also get

$$\bar{\nabla}^a \frac{\delta \mathscr{A}}{\delta h^{ab}} = 0 \tag{2.31}$$

instead of (2.29). We'll refer to this choice of parameters as the "Diff case", while refering to any choice where $a \neq 1$ or $b \neq 1$ as the "TDiff case". A mass term is usually defined as an action term free of derivatives, quadratic in the field that breaks gauge invariance and hence increases the number of propagating DOFs. In the Diff case, this would be

$$m^2 \int d^D x \left(h^2 + k' h^{ab} h_{ab}\right) \tag{2.32}$$

and $k' = -1$ corresponds to the FP mass term. Since the trace h is invariant under TDiff, in this case the form of the mass term is

$$m^2 \int d^D x \, h^{ab} h_{ab} \tag{2.33}$$

2.3 Why self-coupling?

Consider a couple of action funtionals $\mathscr{A}_h$ and $\mathscr{A}_\varphi$ for SRFTs of a free h^{ab} field and a free matter field φ. For completeness, we add to $\mathscr{A}_h$ a mass term $\mathscr{A}_h^{\mathrm{m}}$ like the ones considered in (2.32) and (2.33). The action for a theory where both fields interact should have the following form:

$$\mathscr{A}_{\text{tot}} = \mathscr{A}_h + \mathscr{A}_h^{\mathrm{m}} + \mathscr{A}_\varphi + \chi \mathscr{A}_{\text{int}} \tag{2.34}$$

where χ, in front of the interaction term $\mathscr{A}_{\text{int}} \equiv \mathscr{A}_{\text{int}}[\eta; h, \varphi]$, is a coupling constant. The EOMs are $\frac{\delta \mathscr{A}_{\text{tot}}}{\delta h^{ab}} = 0$ and $\frac{\delta \mathscr{A}_{\text{tot}}}{\delta \varphi} = 0$. Note that

$$\frac{\delta \mathscr{A}_{\text{tot}}}{\delta h^{ab}} = \frac{\delta \mathscr{A}_h}{\delta h^{ab}} + \chi \frac{\delta \mathscr{A}_{\text{int}}^{\mathrm{m}}}{\delta h^{ab}} \tag{2.35}$$

where $\mathscr{A}^{\mathrm{m}}_{\mathrm{int}} \equiv \chi^{-1}\mathscr{A}^{\mathrm{m}}_{h} + \mathscr{A}_{\mathrm{int}}$. Suppose that $\mathscr{A}_h$ is gauge invariant under (2.18). Taking into account section 2.2, if the gauge transformation parameter is arbitrary (Diff symmetry) we have the Gauge/Bianchi Identity

$$\partial^a \frac{\delta \mathscr{A}_h}{\delta h^{ab}} = 0 \tag{2.36}$$

If on the other hand the parameter is contrained by (2.19) (TDiff symmetry), the Gauge/Bianchi Identity is

$$\partial_{[c}\partial^a \frac{\delta \mathscr{A}_h}{\delta h^{b]a}} = 0 \tag{2.37}$$

Using (2.35) together with (2.36) and (2.37) one gets, respectively,

$$\partial^a \frac{\delta \mathscr{A}_{\mathrm{tot}}}{\delta h^{ab}} = \chi \partial^a \frac{\delta \mathscr{A}^{\mathrm{m}}_{\mathrm{int}}}{\delta h^{ab}} \tag{2.38}$$

and

$$\partial_{[c}\partial^a \frac{\delta \mathscr{A}_{\mathrm{tot}}}{\delta h^{b]a}} = \chi \partial_{[c}\partial^a \frac{\delta \mathscr{A}^{\mathrm{m}}_{\mathrm{int}}}{\delta h^{b]a}} \tag{2.39}$$

The r.h.s. of these equations must vanish, when h^{ab} is on-shell. For now, we focus on the Diff case. Bearing in mind the symmetry of $\frac{\delta \mathscr{A}^{m}_{\mathrm{int}}}{\delta h^{ab}}$ and that this together with (2.38) implies on-shell divergencelessness on both indices, it must of the form:

$$\frac{\delta \mathscr{A}^{\mathrm{m}}_{\mathrm{int}}}{\delta h^{ab}} = \partial^c \partial^d \Psi_{[c(a][b)d]}[\eta; h, \varphi] + \mathfrak{B}^{\ \ cd}_{(ab)}[\eta; h, \varphi] \frac{\delta \mathscr{A}_{\mathrm{tot}}}{\delta h^{cd}} \tag{2.40}$$

where $\partial^c \partial^d \Psi_{[c(a][b)d]} = \frac{1}{2}\partial^c \partial^d \left(\Psi_{[ca][bd]} + \Psi_{[cb][ad]} \right)$ is an identically divergenceless term[11]. The second term[12] vanishes on-shell. What about a term that doesn't vanish on-shell but its divergence does? We all know such a term: the EMT. However, conservation (divergencelessness) of the EMT requires all dynamical fields to be on-shell - we've been using this word only w.r.t. the h^{ab} field.

Let us focus on the second term. To avoid unnecessarily long expressions, we make $\Psi = 0$ (this doesn't interfere with the point we're trying to make). Using (2.35), one has

$$\begin{aligned} \frac{\delta \mathscr{A}^{\mathrm{m}}_{\mathrm{int}}}{\delta h^{ab}} &= \mathfrak{B}^{\ \ cd}_{(ab)}[\eta; h, \varphi] \left(\frac{\delta \mathscr{A}_h}{\delta h^{cd}} + \chi \frac{\delta \mathscr{A}^{\mathrm{m}}_{\mathrm{int}}}{\delta h^{cd}} \right) \\ \Leftrightarrow \frac{\delta \mathscr{A}_{\mathrm{int}}}{\delta h^{ab}} &= \frac{\chi^{-1}}{\chi^{-1}\mathfrak{B}^{-1}{}^{(ab)}_{\ \ cd}[\eta; h, \varphi] - \mathbb{1}^{ab}_{cd}} \frac{\delta \mathscr{A}_h}{\delta h^{cd}} - \chi^{-1} \frac{\delta \mathscr{A}^{\mathrm{m}}_{h}}{\delta h^{ab}} \end{aligned} \tag{2.41}$$

where $\mathbb{1}^{ab}_{cd} \equiv \delta^a_c \delta^b_d$. Note that

$$\left(\mathbb{1} - \chi^{-1}\mathfrak{B}^{-1}\right)^{-1} = \mathbb{1} + \chi^{-1}\mathfrak{B}^{-1} + \left(\chi^{-1}\mathfrak{B}^{-1}\right)^2 + \ldots \tag{2.42}$$

[11] A term proportional to $\sqrt{-|\eta|}\eta_{ab}$ can be included in this, which would require Ψ to depend explictely on x^a.

[12] $\mathfrak{B}$ could a priori contain derivatives, but this would lead to higher derivative non-local theories.

We're not interested in the possibility of having negative powers of the coupling constant so we make $\mathfrak{B} = 0$. Then, using (2.40) on (2.35),

$$\partial^a \frac{\delta \mathscr{A}_{\text{tot}}}{\delta h^{ab}} = 0 \tag{2.43}$$

identically. Taking into consideration the process of deriving a Gauge/Bianchi Identity and requiring that the dynamical fields mantain the number of propagating DOFs, this equation is basically telling that the interacting theory must enjoy Diff symmetry (2.18). Chapters 3.2.5 and 3.2.7 of [33] (and references therein, including [15]) shed some light on this possibility. In this work, we're not interested in it so we would like to find a new non-identically divergenceless term.

We turn again our attention to the EMT and we ask: what if the interacting theory we're trying to find turns out to have an EMT that is divergenceless when h^{ab} is on-shell (independently of the matter field)? Assuming this hypothesis, we could write

$$\frac{\delta \mathscr{A}^{\text{m}}_{\text{int}}}{\delta h^{ab}} = T_{ab} + \partial^c \partial^d \Psi_{[c(a][b)d]}[\eta; h, \varphi] \tag{2.44}$$

where to ensure divergenceless on both indices of the EMT we call on Rosenfeld's prescription. Using (2.38), one has that $\partial^a \frac{\delta \mathscr{A}_{\text{tot}}}{\delta h^{ab}} = \chi \partial^a T_{ab}$, showing that our hypothesis is self-consistent. Now, it's only a matter of finding the interaction term that solves (2.44). Lastly note that this equation together with (2.35) entail a coupling of the kind

$$\frac{\delta \mathscr{A}_{\text{tot}}}{\delta h^{ab}} = \frac{\delta \mathscr{A}_h}{\delta h^{ab}} + \chi \left(T_{ab} + \partial^c \partial^d \Psi_{[c(a][b)d]}[\eta; h, \varphi] \right) \tag{2.45}$$

which includes *self-coupling* (since T_{ab} is the total EMT, i.e. it includes contributions from the h^{ab} field itself).

In the case of TDiff, for which not (2.38) but (2.39) is of interest, the differences are minor. In this case, we can additionally have terms in (2.44) whose divergence is identically a gradient: $\eta_{ab}\rho$ and $\partial_a \partial_b \tilde{\rho}$. Besides, like the EMT verifies $\partial^a T_{ab}$, there may be an object X_{ab} $(= X_{ba})$ such that $\partial_{[c} \partial^a X_{b]a} = 0$, on-shell. This amounts to

$$\frac{\delta \mathscr{A}^{\text{m}}_{\text{int}}}{\delta h^{ab}} = X_{ab} + \partial^c \partial^d \Psi_{[c(a][b)d]}[\eta; h, \varphi] + \eta_{ab}\rho[\eta; h, \varphi] + \partial_a \partial_b \tilde{\rho}[\eta; h, \varphi] \tag{2.46}$$

3 Energy-momentum tensors

The EMT emerged in section 2.3 as something "that doesn't vanish on-shell but its divergence does". Here we will elaborate on this object, particularly in the context of *covariant field theories* (much of this chapter is inspired in [34]; [35] and [36] are also pertinent). We then naturally translate our analysis to "U-covariant field theories" (defined in section 3.3) and end up finding the object X_{ab} introduced in the previous paragraph.

3.1 Field theories

Consider an elementary one-dimensional mechanical system described by a lagrangian

$$L = L\Big(y(t), \dot{y}(t), t\Big) \ . \tag{3.1}$$

The action is $S = \int_{t_i}^{t_f} dt\, L$ and the EOM is[13]

$$\frac{\delta S}{\delta y} = 0 \tag{3.2}$$

where $\frac{\delta S}{\delta y} \equiv \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}}$. The lagrangian is defined up to terms which have no effect over the EOM, such as total derivatives and constant terms. These contribute to the action, respectively, with boundary terms and integrals that don't depend on y, $\dot{y}$ or t.

If L depends on the parameter t solely through the dynamical variable y and its derivative $\dot{y}$, the EOM implies

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\dot{y} - L\right) = 0 \tag{3.3}$$

since

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\dot{y} - L\right) = -\frac{\delta S}{\delta y}\dot{y} \tag{3.4}$$

Equation (3.3) enforces the conservation of quantity $\frac{\partial L}{\partial \dot{y}}\dot{y} - L$ that we call energy. A total derivative term in the lagrangian has a null contribution to $\frac{\partial L}{\partial \dot{y}}\dot{y} - L$. On the other side a constant term enters this expression directly through the $-L$ term. One could call the quantity $\frac{\partial L}{\partial \dot{y}}\dot{y} - L + c$ energy, where c is a constant, and the equation of motion would still be satisfied.

Having seen in a simple settting how the notion of energy comes up, let's move on to the more useful context of field theories, not necessarily SRFTs. Consider the action $S = \int d^D x\, \mathscr{L}$ for a

[13] Considering vanishing δy at the limits of integration.

generic collection of different types of fields. Assume the lagrangian has no explicit dependence on x^a: $\mathscr{L} = \mathscr{L}\left[\varphi^\alpha(x), \partial_a\varphi^\alpha(x)\right]$. The EOMs are

$$\frac{\delta S}{\delta\varphi^\alpha} = 0 \tag{3.5}$$

where $\frac{\delta S}{\delta\varphi^\alpha} \equiv \frac{\partial\mathscr{L}}{\partial\varphi^\alpha} - \partial_a\left(\frac{\partial\mathscr{L}}{\partial\partial_a\varphi^\alpha}\right)$. An on-shell conservation law like (3.3) is obtained:

$$\partial_b\left(\delta^b_a\mathscr{L} - \frac{\partial\mathscr{L}}{\partial\partial_b\varphi^\alpha}\partial_a\varphi^\alpha\right) = \frac{\delta S}{\delta\varphi^\alpha}\partial_a\varphi^\alpha \tag{3.6}$$

(using commutativity of ∂). We call

$$t_{\text{Can}}{}^b{}_a = \frac{\partial\mathscr{L}}{\partial\partial_b\varphi^\alpha}\partial_a\varphi^\alpha - \delta^b_a\mathscr{L} \tag{3.7}$$

the canonical energy-momentum "tensor" (there's no need to worry about coordinate transformations for now). Analagously to the addition of a constant c to the energy above, $t_{\text{Can}}{}^b{}_a$ is defined up to off-shell divergenceless terms $\Omega^b{}_a$, i.e. $\partial_b\Omega^b{}_a = 0$ identically[14].

3.2 Canonical and Rosenfeld's EMTs

Let us focus on covariant field theories with a (2,0) tensor field h with symmetric components[15] h^{ab} and others of arbitrary tensor ranks, collectively denoted by φ, as dynamical fields. We choose to place them in a flat spacetime (in this and the following sections, this is only used for the conservation of the canonical EMT; all other results are independent of the covariant derivatives commuting or not) with

- the metric $\bar{g}$ whose components $\bar{g}_{ab}(x)$ are written in arbitrary curvilinear coordinates x^a,
- and the covariant derivative $\bar{\nabla}$ of the Levi-Civita connection $\Gamma[\bar{g}]$ (recall that the Christoffel symbols $\Gamma^c_{ab}(x)$ are built from the metric and its derivatives).

Note that this is exactly the kind of theory that could be attained through the "covariantization" described in section 2.1, starting with a SRFT where the dynamical fields are the Lorentz (instead of spacetime) tensors h^{ab} and φ.

Covariance of the theory, i.e. of the EOMs, is obtained by having an action $S = \int d^Dx\,\mathscr{L}$ that is invariant under GCTs, i.e. a scalar. The lagrangian is thus required to be a weight-1

[14]One can choose $\Omega^b{}_a$ in order to make $t_{\text{Can}}{}^b{}_a$ symmetric under $a \leftrightarrow b$.

[15]We always refer to components in the coordinate basis.

scalar density, namely $\mathscr{L} = \sqrt{-|\bar{g}|}\mathcal{L}$ where $\mathcal{L}$ is a scalar formed by summing products of contractions between components of the dynamical tensor fields, its covariant derivatives and $\bar{g}$. We'll start with the following choice for $\mathscr{L}$'s independent variables:

$$\mathscr{L} = \mathscr{L}[\bar{g}^{ab}, \partial_c \bar{g}^{ab}; h^{ab}, \partial_c h^{ab}, \varphi, \partial_c \varphi] \tag{3.8}$$

Since $\bar{g}^{ab}$ and $\partial_c \bar{g}^{ab}$ are not dynamical variables, they make $\mathscr{L}$ depend on x^a explicitly. From this, one does not expect to obtain on-shell conservation laws, at least not in terms of vanishing coordinate derivatives. The EOMs are

$$\begin{cases} 0 = \frac{\partial \mathscr{L}}{\partial h^{ab}} - \partial_c \left(\frac{\partial \mathscr{L}}{\partial \partial_c h^{ab}} \right) \equiv \frac{\delta S}{\delta h^{ab}} \\ 0 = \frac{\partial \mathscr{L}}{\partial \varphi} - \partial_c \left(\frac{\partial \mathscr{L}}{\partial \partial_c \varphi} \right) \equiv \frac{\delta S}{\delta \varphi} \end{cases} \tag{3.9}$$

Writing h^{ab} and φ as ϕ^α with $\alpha = 1$ and 2, respectively, such that $\mathscr{L}[\bar{g}^{ab}, \partial_c \bar{g}^{ab}; h^{ab}, \partial_c h^{ab}, \varphi, \partial_c \varphi] \equiv \mathscr{L}[\bar{g}^{ab}, \partial_c \bar{g}^{ab}; \phi^\alpha, \partial_c \phi^\alpha]$, we have

$$\partial_c \mathscr{L} = \frac{\partial \mathscr{L}}{\partial \phi^\alpha} \partial_c \phi^\alpha + \frac{\partial \mathscr{L}}{\partial \partial_d \phi^\alpha} \partial_c \partial_d \phi^\alpha + \frac{\partial \mathscr{L}}{\partial \bar{g}^{ab}} \partial_c \bar{g}^{ab} + \frac{\partial \mathscr{L}}{\partial \partial_d \bar{g}^{ab}} \partial_c \partial_d \bar{g}^{ab} \, . \tag{3.10}$$

Thus we see that

$$\begin{aligned} &\partial_d \left(\delta^d_c \mathscr{L} - \frac{\partial \mathscr{L}}{\partial \partial_d \phi^\alpha} \partial_c \phi^\alpha \right) = \frac{\delta S}{\delta \phi^\alpha} \partial_c \phi^\alpha + \frac{\partial \mathscr{L}}{\partial \bar{g}^{ab}} \partial_c \bar{g}^{ab} + \frac{\partial \mathscr{L}}{\partial \partial_d \bar{g}^{ab}} \partial_c \partial_d \bar{g}^{ab} \\ \Leftrightarrow &\partial_d \left(\delta^d_c \mathscr{L} - \frac{\partial \mathscr{L}}{\partial \partial_d \phi^\alpha} \partial_c \phi^\alpha - \frac{\partial \mathscr{L}}{\partial \partial_d \bar{g}^{ab}} \partial_c \bar{g}^{ab} \right) = \frac{\delta S}{\delta \phi^\alpha} \partial_c \phi^\alpha + \left[\frac{\partial \mathscr{L}}{\partial \bar{g}^{ab}} - \partial_d \left(\frac{\partial \mathscr{L}}{\partial \partial_d \bar{g}^{ab}} \right) \right] \partial_c \bar{g}^{ab} \end{aligned} \tag{3.11}$$

but the last term is not an equation of motion so it will not vanish on-shell, confirming our expectation regarding conservation laws. However, due to metric compatibility, a generalisation of the conservation laws in terms of vanishing covariant derivatives is available. As [34] points out, *this is consistent with our ability to choose coordinates x^a in such a way that $\bar{g}^{ab}$ will become a constant matrix and Γ^c_{ab} will all vanish, thus removing the explicit dependence of $\mathscr{L}$ on coordinates*. Now, we choose alternatively the following $\mathscr{L}$'s independent variables:

$$\mathscr{L} \equiv \tilde{\mathscr{L}} = \tilde{\mathscr{L}}[\bar{g}^{ab}; h^{ab}, \bar{\nabla}_c h^{ab}, \varphi, \bar{\nabla}_c \varphi] \tag{3.12}$$

Taking into account commutativity between variation and covariant derivative, one arrives at the EOMs in an explicitly covariant form:

$$\begin{cases} 0 = \frac{\partial \tilde{\mathscr{L}}}{\partial h^{ab}} - \bar{\nabla}_c \left(\frac{\partial \tilde{\mathscr{L}}}{\partial \bar{\nabla}_c h^{ab}} \right) \\ 0 = \frac{\partial \tilde{\mathscr{L}}}{\partial \varphi} - \bar{\nabla}_c \left(\frac{\partial \tilde{\mathscr{L}}}{\partial \bar{\nabla}_c \varphi} \right) \end{cases} \tag{3.13}$$

Note that

$$\begin{aligned}\frac{\partial\tilde{\mathscr{L}}}{\partial h^{ab}}-\bar{\nabla}_c\left(\frac{\partial\tilde{\mathscr{L}}}{\partial\bar{\nabla}_c h^{ab}}\right)&=\frac{\partial\tilde{\mathscr{L}}}{\partial h^{ab}}-\bar{\nabla}_c\left(\frac{\partial\mathscr{L}}{\partial\partial_c h^{ab}}\right)\\&=\frac{\partial\tilde{\mathscr{L}}}{\partial h^{ab}}-\partial_c\left(\frac{\partial\mathscr{L}}{\partial\partial_c h^{ab}}\right)-\Gamma^c_{cd}\left(\frac{\partial\mathscr{L}}{\partial\partial_d h^{ab}}\right)+2\Gamma^d_{ca}\left(\frac{\partial\mathscr{L}}{\partial\partial_c h^{db}}\right)\\&=\frac{\partial\mathscr{L}}{\partial h^{ab}}-\partial_c\left(\frac{\partial\mathscr{L}}{\partial\partial_c h^{ab}}\right)=\frac{\delta S}{\delta h^{ab}}\ .\end{aligned}\tag{3.14}$$

and similar for φ. Writing h^{ab} and φ as ϕ^α with $\beta=1$ and 2, respectively, such that $\tilde{\mathscr{L}}[\bar{g}^{ab};h^{ab},\bar{\nabla}_c h^{ab},\varphi,\bar{\nabla}_c\varphi]=\tilde{\mathscr{L}}[\bar{g};\phi^\beta,\bar{\nabla}_a\phi^\beta]$,

$$\begin{aligned}\bar{\nabla}_c\tilde{\mathcal{L}}=\partial_c\tilde{\mathcal{L}}&=\frac{\partial\tilde{\mathcal{L}}}{\partial\phi^\beta}\partial_c\phi^\beta+\frac{\partial\tilde{\mathcal{L}}}{\partial\bar{\nabla}_d\phi^\beta}\partial_c\bar{\nabla}_d\phi^\beta+\frac{\partial\tilde{\mathcal{L}}}{\partial\bar{g}_{ab}}\partial_c\bar{g}_{ab}\\&=\frac{\partial\tilde{\mathcal{L}}}{\partial\phi^\beta}\bar{\nabla}_c\phi^\beta+\frac{\partial\tilde{\mathcal{L}}}{\partial\bar{\nabla}_d\phi^\beta}\bar{\nabla}_c\bar{\nabla}_d\phi^\beta+\frac{\partial\tilde{\mathcal{L}}}{\partial\bar{g}_{ab}}\bar{\nabla}_c\bar{g}_{ab}\end{aligned}\tag{3.15}$$

The last equality[16] may seem a big step but is actually quite natural once one takes into consideration that $\tilde{\mathcal{L}}$ is formed only by tensors ϕ, $\bar{\nabla}\phi$ and g and these are all contracted, so if one expands $\bar{\nabla}_c$ in the last line of (3.15) half of the terms with Γ cancel out the other half. Using metric compatibility,

$$\begin{aligned}\bar{\nabla}_c\tilde{\mathcal{L}}&=\frac{\partial\tilde{\mathcal{L}}}{\partial\phi^\beta}\bar{\nabla}_c\phi^\beta+\frac{\partial\tilde{\mathcal{L}}}{\partial\bar{\nabla}_d\phi^\beta}\bar{\nabla}_c\bar{\nabla}_d\phi^\beta\\\Rightarrow\bar{\nabla}_c\tilde{\mathscr{L}}&=\frac{\partial\tilde{\mathscr{L}}}{\partial\phi^\beta}\bar{\nabla}_c\phi^\beta+\frac{\partial\tilde{\mathscr{L}}}{\partial\bar{\nabla}_d\phi^\beta}\bar{\nabla}_c\bar{\nabla}_d\phi^\beta\end{aligned}\tag{3.16}$$

and we have (using $\bar{\nabla}$'s commutativity)

$$\frac{\delta S}{\delta\phi^\beta}\bar{\nabla}_c\phi^\beta=\bar{\nabla}_d\left(\delta^d_c\tilde{\mathscr{L}}-\frac{\partial\tilde{\mathscr{L}}}{\partial\bar{\nabla}_d\phi^\beta}\bar{\nabla}_c\phi^\beta\right)\ .\tag{3.17}$$

We finally arrived at the on-shell covariant conservation law[17]

$$\frac{\delta S}{\delta h^{ab}}\bar{\nabla}_c h^{ab}+\frac{\delta S}{\delta\varphi}\bar{\nabla}_c\varphi=\bar{\nabla}_d\left(\delta^d_c\tilde{\mathscr{L}}-\frac{\partial\tilde{\mathscr{L}}}{\partial\bar{\nabla}_d h^{ab}}\bar{\nabla}_c h^{ab}-\frac{\partial\tilde{\mathscr{L}}}{\partial\bar{\nabla}_d\varphi}\bar{\nabla}_c\varphi\right)\tag{3.18}$$

[16]which takes the form of a chain rule with the covariant derivative.

[17]This is not a conservation law in the strict sense but we follow [35] in calling *covariant conservation law any covariant relation that reduces, in the flat limit, to a conservation law.*

such that the canonical EMT is

$$\begin{aligned}T^{cd}_{\text{Can}} &= \frac{1}{\sqrt{-|\bar{g}|}}\left(\frac{\partial\tilde{\mathscr{L}}}{\partial\bar{\nabla}_d h^{ab}}\bar{\nabla}_e h^{ab} + \frac{\partial\tilde{\mathscr{L}}}{\partial\bar{\nabla}_d\varphi}\bar{\nabla}_e\varphi - \delta^d_e\tilde{\mathscr{L}}\right)\bar{g}^{ce}\\ &= \frac{1}{\sqrt{-|\bar{g}|}}\left(\frac{\partial\mathscr{L}}{\partial\partial_d h^{ab}}\bar{\nabla}_e h^{ab} + \frac{\partial\mathscr{L}}{\partial\partial_d\varphi}\bar{\nabla}_e\varphi - \delta^d_e\mathscr{L}\right)\bar{g}^{ce}\end{aligned} \tag{3.19}$$

One can derive other EM tensor - Rosenfeld's [20] - by taking the active transformation perpective on infinitesimal GCTs and using gauge invariance associated with it. An infinitesimal GCT

$$x^a \to x'^a = x^a - \xi^a(x) \tag{3.20}$$

generates a transformation of the fields given by the Lie derivative $\mathcal{L}_\xi$ with respect to the vector field ξ^a. For a weight-ω (n,m)-tensor density (an ordinary tensor has $\omega = 0$),

$$\begin{aligned}\mathcal{L}_\xi T^{a_1\ldots a_n}{}_{b_1\ldots b_m} =&\xi^c\partial_c T^{a_1\ldots a_n}{}_{b_1\ldots b_m} - T^{c\ldots a_n}{}_{b_1\ldots b_m}\partial_c\xi^{a_1} - \ldots\\ &+ T^{a_1\ldots a_n}{}_{c\ldots b_m}\partial_{b_1}\xi^c + \ldots + \omega T^{a_1\ldots a_n}{}_{b_1\ldots b_m}\partial_c\xi^c\\ =&\xi^c\bar{\nabla}_c T^{a_1\ldots a_n}{}_{b_1\ldots b_m} - T^{c\ldots a_n}{}_{b_1\ldots b_m}\bar{\nabla}_c\xi^{a_1} - \ldots\\ &+ T^{a_1\ldots a_n}{}_{c\ldots b_m}\bar{\nabla}_{b_1}\xi^c + \ldots + \omega T^{a_1\ldots a_n}{}_{b_1\ldots b_m}\bar{\nabla}_c\xi^c\ .\end{aligned} \tag{3.21}$$

where the second equality is allowed for any covariant derivative associated with a torsion-free connection (not necessarily flat). Since $\mathscr{L}$ is a weight-1 scalar density,

$$\delta\mathscr{L} = \partial_a\left(\xi^a\mathscr{L}\right) \tag{3.22}$$

and the action changes by a total derivative, leading to gauge invariance with respect to infinitesimal GCT. We are going to compare (3.22) with $\delta\mathscr{L}$ caused by an arbitrary variation of $\bar{g}^{ab}$, h^{ab} and φ. Then we will enforce the EOMs and use $\delta\bar{g}^{ab}$ for a infinitesimal GCT explicitly:

$$\delta\bar{g}^{ab} = \xi^c\bar{\nabla}_c\bar{g}^{ab} - 2\bar{g}^{c(a}\bar{\nabla}_c\xi^{b)} = -2\bar{\nabla}^{(a}\xi^{b)} \tag{3.23}$$

(using the symmetry of $\bar{g}$'s components and $\bar{\nabla}$'s metric compatibility). We have

$$\partial_a\left(\xi^a\mathscr{L}\right) = \frac{\delta S}{\delta h^{ab}}\delta h^{ab} + \frac{\delta S}{\delta\varphi}\delta\varphi + \frac{\delta S}{\delta\bar{g}^{ab}}\delta\bar{g}^{ab} + \partial_c\left(\frac{\partial\mathscr{L}}{\partial\partial_c h^{ab}}\delta h^{ab} + \frac{\partial\mathscr{L}}{\partial\partial_c\varphi}\delta\varphi + \frac{\partial\mathscr{L}}{\partial\partial_c\bar{g}^{ab}}\delta\bar{g}^{ab}\right) \tag{3.24}$$

where $\frac{\delta S}{\delta\bar{g}^{ab}} \equiv \frac{\partial\mathscr{L}}{\partial\bar{g}^{ab}} - \partial_c\left(\frac{\partial\mathscr{L}}{\partial\partial_c\bar{g}^{ab}}\right)$. Using (3.9) and (3.23) first and then integrating this by parts, one has

$$2\int d^Dx\,\xi^b\bar{\nabla}^a\frac{\delta S}{\delta\bar{g}^{ab}} + ST = 0 \tag{3.25}$$

The surface term can be converted into an integral over the boundary through Stokes' Theorem and using an appropriate choice of boundary conditions for the transformation parameter ξ it can be made to vanish. Due to arbitrariness of ξ one arrives at the following on-shell covariant conservation law:

$$\bar{\nabla}^a \frac{\delta S}{\delta \bar{g}^{ab}} = 0 \tag{3.26}$$

(Note that $\frac{\delta S}{\delta \bar{g}_{cd}} = -\bar{g}^{ac}\bar{g}^{bd}\frac{\delta S}{\delta \bar{g}^{ab}}$). If we fix inertial coordinates we get the on-shell conservation law that we were looking for T_{ab} in (2.44): $\partial^a \frac{\delta S}{\delta \bar{g}^{ab}}\Big|_{\bar{g}=\eta} = 0 = \partial^b \frac{\delta S}{\delta \bar{g}^{ab}}\Big|_{\bar{g}=\eta}$. However, motivated by the covariant nature of this object, we decide from now on to consider the covariantizations S_h, S_h^{m}, S_φ and S_{int} of actions $\mathscr{A}_h$ (invariant under Diff), $\mathscr{A}_h^{\mathrm{m}}$, $\mathscr{A}_\varphi$ and $\mathscr{A}_{\mathrm{int}}$ of section 2.3, respectively. Hence, we rewrite (2.44) as

$$\frac{\delta S_{\mathrm{int}}^{\mathrm{m}}}{\delta h^{ab}} = \frac{\delta S_{\mathrm{tot}}}{\delta \bar{g}^{ab}} + \bar{\nabla}^c\bar{\nabla}^d \Psi_{[c(a][b)d]}[\bar{g}; h, \varphi] \tag{3.27}$$

$$\Leftrightarrow \bar{\nabla}^a \frac{\delta S_{\mathrm{int}}^{\mathrm{m}}}{\delta h^{ab}} = \bar{\nabla}^a \frac{\delta S_{\mathrm{tot}}}{\delta \bar{g}^{ab}} \tag{3.28}$$

where $S_{\mathrm{int}}^{\mathrm{m}} \equiv \chi^{-1} S_h^{\mathrm{m}} + S_{\mathrm{int}}$ and $S_{\mathrm{tot}} \equiv S_h + S_\varphi + \chi S_{\mathrm{int}}^{\mathrm{m}}$. All the r.h.s. of (3.27) is an EMT (density) since this quantity is defined up to identically divergenceless terms. Therefore our "consistent coupling" is equivalent to the one in most of the literature mentioned in the introduction, where these terms weren't often made explicit. For completeness, we write the Rosenfeld's EMT

$$T^{ab}_{\mathrm{Ros}} \equiv \frac{2}{\sqrt{-|\bar{g}|}} \frac{\delta S}{\delta \bar{g}^{cd}} \bar{g}^{ac}\bar{g}^{bd} = \frac{-2}{\sqrt{-|\bar{g}|}} \frac{\delta S}{\delta \bar{g}_{ab}} \; . \tag{3.29}$$

Note that the covariant conservation law (3.26) is valid independently of the metric being flat or curved (its derivation didn't required commutativity of covariant derivatives, contrasting with the canonical EMT). One can now ask: what is the relation between the canonical and Rosenfeld's EMTs? The answer comes via the Gauge/Bianchi Identity associated with infinitesimal GCT. To derive this, we'll use the variations

$$\delta h^{ab} = \xi^c \bar{\nabla}_c h^{ab} - 2h^{c(a}\bar{\nabla}_c \xi^{b)} \tag{3.30}$$

and

$$\delta\varphi = \xi^c \bar{\nabla}_c \varphi \; , \tag{3.31}$$

under GCTs (for concreteness, we suppose that φ is a scalar). Integrating (3.24), one obtains

$$\int d^D x \left(\frac{\delta S}{\delta h^{ab}} \delta h^{ab} + \frac{\delta S}{\delta \varphi} \delta\varphi + \frac{\delta S}{\delta \bar{g}^{ab}} \delta \bar{g}^{ab} \right) + ST = 0 \tag{3.32}$$

Using (3.23), (3.30) and (3.31) on the equation above:

$$\begin{aligned}
&\int d^D x \left(-2\frac{\delta S}{\delta h^{ab}} h^{ca} \bar{\nabla}_c \xi^b + \frac{\delta S}{\delta h^{ab}} \xi^c \bar{\nabla}_c h^{ab} + \frac{\delta S}{\delta \varphi} \xi^c \bar{\nabla}_c \varphi - 2\frac{\delta S}{\delta \bar{g}^{ab}} \bar{\nabla}^a \xi^b \right) + ST \\
&= \int d^D x \left(-2\frac{\delta S}{\delta h^{ab}} h^{ca} \bar{\nabla}_c \xi^b - \sqrt{-|\bar{g}|} \xi^c \bar{\nabla}_d T^{ed}_{\text{Can}} \bar{g}_{ec} - 2\frac{\delta S}{\delta \bar{g}^{ab}} \bar{\nabla}^a \xi^b \right) + ST = 0
\end{aligned} \tag{3.33}$$

Integrating by parts,

$$\int d^D x \left[2\bar{\nabla}_c \left(\frac{\delta S}{\delta h^{ab}} h^{ac} \right) - \sqrt{-|\bar{g}|} \bar{\nabla}_c T^{ec}_{\text{Can}} \bar{g}_{eb} + 2\bar{\nabla}^a \frac{\delta S}{\delta \bar{g}^{ab}} \right] \xi^b + ST = 0 \tag{3.34}$$

The last term can be converted into an integral over the boundary and, choosing appropriate boundary conditions for the transformation parameters, it can be made to vanish. Due to arbitrariness of ξ one obtains the Gauge/Bianchi Identity, which don't involve the transformation parameters:

$$\bar{\nabla}_c \left(2\frac{\delta S}{\delta h^{ab}} h^{ac} - \sqrt{-|\bar{g}|} T^{ec}_{\text{Can}} g_{eb} + 2\frac{\delta S}{\delta \bar{g}^{ab}} \bar{g}^{ac} \right) = 0\ . \tag{3.35}$$

Since this is an off-shell identity and using the definition of T^{ab}_{Ros}, one arrives at

$$\begin{aligned}
&\bar{\nabla}_c \left(2\frac{\delta S}{\delta h^{ab}} h^{ac} - \sqrt{-|\bar{g}|} T^{ec}_{\text{Can}} g_{eb} + \sqrt{-|\bar{g}|} T^{ec}_{\text{Ros}} g_{eb} \right) = 0 \\
&\Rightarrow 2\frac{\delta S}{\delta h^{ab}} h^{ac} - \sqrt{-|\bar{g}|} T^{ec}_{\text{Can}} g_{eb} + \sqrt{-|\bar{g}|} T^{ec}_{\text{Ros}} g_{eb} = \sqrt{-|\bar{g}|} \bar{\nabla}_d {\psi_b}^{cd}
\end{aligned} \tag{3.36}$$

where the r.h.s. is identically divergenceless (${\psi_b}^{cd} = -{\psi_b}^{dc}$ is a superpotential). In appendix B, we present an alternative derivation of this relation between the EMTs (at the expense of easiness, we manage to find the superpotential explicitly).

It will turn up useful to write (3.35) as

$$-2\bar{g}^{ac} \bar{\nabla}_c \frac{\delta S}{\delta \bar{g}^{ab}} = \frac{\delta S}{\delta h^{ac}} \bar{\nabla}_b h^{ac} + \frac{\delta S}{\delta \varphi} \bar{\nabla}_b \varphi + 2\bar{\nabla}_c \left(\frac{\delta S}{\delta h^{ab}} h^{ac} \right) \tag{3.37}$$

3.3 U-covariant Field Theories

In this section, (following what we did in p. 8) we will write "U-" as a prefix meaning that a representation of UCTs, the one used in section 2.1 (which is a subrepresentation of the usual one for GCTs), is implied. Since, except for scalar ones, all tensor densitites we're going to deal with have zero weight, we might omit the prefix in these cases.

Let us focus on U-covariant field theories with the same dynamical fields as the covariant theories considered in 3.2, except that they're U-tensors. They also live in a flat spacetime but this time

- the metric's components $\bar{\mathfrak{g}}_{ab}(x)$ are written in unimodular (otherwise arbitrary) curvilinear coordinates x^a,
- and the covariant derivative $\bar{\boldsymbol{\nabla}}$ is associated to the Levi-Civita connection $\boldsymbol{\Gamma}[\bar{\mathfrak{g}}]$.

Note that, since we're using unimodular coordinates, $\boldsymbol{\Gamma}^b_{ba} = 0$. This is the kind of theory that we get when a SRFT is "U-covariantized", as described in section 2.1.

U-covariance of the theory is attained by having a U-scalar action $S_U = \int d^D x \, \mathscr{L}_U$. Then, the lagrangian is a U-scalar (since $\int d^D x$ also is), formed by summing products of contractions between components of the dynamical tensor fields, its covariant derivatives and $\bar{\mathfrak{g}}$. Once again, we must fix a set of $\mathscr{L}_U$'s independent variables. By performing the choice equivalent to (3.8)

$$\mathscr{L}_U = \mathscr{L}_U[\bar{\mathfrak{g}}^{ab}, \partial_c \bar{\mathfrak{g}}^{ab}; h^{ab}, \partial_c h^{ab}, \varphi, \partial_c \varphi] \tag{3.38}$$

we would reach the same conclusions. We then start with

$$\mathscr{L}_U \equiv \tilde{\mathscr{L}}_U = \tilde{\mathscr{L}}_U[\bar{\mathfrak{g}}^{ab}; h^{ab}, \bar{\boldsymbol{\nabla}}_c h^{ab}, \varphi, \bar{\boldsymbol{\nabla}}_c \varphi] \tag{3.39}$$

Taking into account commutativity between variation and covariant derivative, one arrives at the EOMs:

$$\begin{cases} 0 = \frac{\partial \tilde{\mathscr{L}}_U}{\partial h^{ab}} - \bar{\boldsymbol{\nabla}}_c \left(\frac{\partial \tilde{\mathscr{L}}_U}{\partial \bar{\boldsymbol{\nabla}}_c h^{ab}} \right) = \frac{\partial \mathscr{L}_U}{\partial h^{ab}} - \partial_c \left(\frac{\partial \mathscr{L}_U}{\partial \partial_c h^{ab}} \right) \equiv \frac{\delta S_U}{\delta h^{ab}} \\ 0 = \frac{\partial \tilde{\mathscr{L}}_U}{\partial \varphi} - \bar{\boldsymbol{\nabla}}_c \left(\frac{\partial \tilde{\mathscr{L}}_U}{\partial \bar{\boldsymbol{\nabla}}_c \varphi} \right) = \frac{\partial \mathscr{L}_U}{\partial \varphi} - \partial_c \left(\frac{\partial \mathscr{L}_U}{\partial \partial_c \varphi} \right) \equiv \frac{\delta S_U}{\delta \varphi} \end{cases} \tag{3.40}$$

Writing h^{ab} and φ as ϕ^α with $\beta = 1$ and 2, respectively, such that $\tilde{\mathscr{L}}_U[\bar{\mathfrak{g}}^{ab}; h^{ab}, \bar{\boldsymbol{\nabla}}_c h^{ab}, \varphi, \bar{\boldsymbol{\nabla}}_c \varphi] = \tilde{\mathscr{L}}_U[\bar{\mathfrak{g}}; \phi^\beta, \bar{\boldsymbol{\nabla}}_a \phi^\beta]$,

$$\begin{aligned} \bar{\boldsymbol{\nabla}}_c \tilde{\mathscr{L}}_U = \partial_c \tilde{\mathscr{L}}_U &= \frac{\partial \tilde{\mathscr{L}}_U}{\partial \phi^\beta} \partial_c \phi^\beta + \frac{\partial \tilde{\mathscr{L}}_U}{\partial \bar{\boldsymbol{\nabla}}_d \phi^\beta} \partial_c \bar{\boldsymbol{\nabla}}_d \phi^\beta + \frac{\partial \tilde{\mathscr{L}}_U}{\partial \bar{\mathfrak{g}}_{ab}} \partial_c \bar{\mathfrak{g}}_{ab} \\ &= \frac{\partial \tilde{\mathscr{L}}_U}{\partial \phi^\beta} \bar{\boldsymbol{\nabla}}_c \phi^\beta + \frac{\partial \tilde{\mathscr{L}}_U}{\partial \bar{\boldsymbol{\nabla}}_d \phi^\beta} \bar{\boldsymbol{\nabla}}_c \bar{\boldsymbol{\nabla}}_d \phi^\beta + \frac{\partial \tilde{\mathscr{L}}_U}{\partial \bar{\mathfrak{g}}_{ab}} \bar{\boldsymbol{\nabla}}_c \bar{\mathfrak{g}}_{ab} \end{aligned} \tag{3.41}$$

This is analogous to (3.15), with the difference that there Γ^b_{ba} didn't vanish and so the first equality above wouldn't be valid with the lagrangian $\tilde{\mathscr{L}}$ (we had to use the scalar $\tilde{\mathcal{L}}$). Using metric compatibility,

$$\bar{\boldsymbol{\nabla}}_c \tilde{\mathscr{L}}_U = \frac{\partial \tilde{\mathscr{L}}_U}{\partial \phi^\beta} \bar{\boldsymbol{\nabla}}_c \phi^\beta + \frac{\partial \tilde{\mathscr{L}}_U}{\partial \bar{\boldsymbol{\nabla}}_d \phi^\beta} \bar{\boldsymbol{\nabla}}_c \bar{\boldsymbol{\nabla}}_d \phi^\beta \tag{3.42}$$

and we have (using $\bar{\boldsymbol{\nabla}}$'s commutativity)

$$\frac{\delta S_U}{\delta\phi^\beta}\bar{\boldsymbol{\nabla}}_c\phi^\beta = \bar{\boldsymbol{\nabla}}_d\left(\delta^d_c\tilde{\mathscr{L}}_U - \frac{\partial\tilde{\mathscr{L}}_U}{\partial\bar{\boldsymbol{\nabla}}_d\phi^\beta}\bar{\boldsymbol{\nabla}}_c\phi^\beta\right)\ . \tag{3.43}$$

We finally arrived at the on-shell covariant conservation law

$$\frac{\delta S_U}{\delta h^{ab}}\bar{\boldsymbol{\nabla}}_c h^{ab} + \frac{\delta S_U}{\delta\varphi}\bar{\boldsymbol{\nabla}}_c\varphi = \bar{\boldsymbol{\nabla}}_d\left(\delta^d_c\tilde{\mathscr{L}}_U - \frac{\partial\tilde{\mathscr{L}}_U}{\partial\bar{\boldsymbol{\nabla}}_d h^{ab}}\bar{\boldsymbol{\nabla}}_c h^{ab} - \frac{\partial\tilde{\mathscr{L}}_U}{\partial\bar{\boldsymbol{\nabla}}_d\varphi}\bar{\boldsymbol{\nabla}}_c\varphi\right) \tag{3.44}$$

such that the canonical EMT is

$$\begin{aligned} T^{cd}_{\text{Can}} &= \left(\frac{\partial\tilde{\mathscr{L}}_U}{\partial\bar{\boldsymbol{\nabla}}_d h^{ab}}\bar{\boldsymbol{\nabla}}_e h^{ab} + \frac{\partial\tilde{\mathscr{L}}_U}{\partial\bar{\boldsymbol{\nabla}}_d\varphi}\bar{\boldsymbol{\nabla}}_e\varphi - \delta^d_e\tilde{\mathscr{L}}_U\right)\bar{\mathfrak{g}}^{ce} \\ &= \left(\frac{\partial\mathscr{L}_U}{\partial\partial_d h^{ab}}\bar{\boldsymbol{\nabla}}_e h^{ab} + \frac{\partial\mathscr{L}_U}{\partial\partial_d\varphi}\bar{\boldsymbol{\nabla}}_e\varphi - \delta^d_e\mathscr{L}_U\right)\bar{\mathfrak{g}}^{ce} \end{aligned} \tag{3.45}$$

Next, we look over the consequences of gauge invariance associated with taking the active transformation perpective on UCTs: an infinitesimal UCT

$$x^a \to x'^a = x^a - \boldsymbol{\xi}^a(x) \quad \text{with} \quad \partial_a\boldsymbol{\xi}^a = \bar{\boldsymbol{\nabla}}_a\boldsymbol{\xi}^a = 0 \tag{3.46}$$

generates a transformation of the fields given by

$$\begin{aligned} \mathcal{L}_\xi T^{a_1\ldots a_n}{}_{b_1\ldots b_m}\Big|_{\xi=\boldsymbol{\xi}^a} =& \boldsymbol{\xi}^c\partial_c T^{a_1\ldots a_n}{}_{b_1\ldots b_m} - T^{c\ldots a_n}{}_{b_1\ldots b_m}\partial_c\boldsymbol{\xi}^{a_1} - \ldots \\ &+ T^{a_1\ldots a_n}{}_{c\ldots b_m}\partial_{b_1}\boldsymbol{\xi}^c + \ldots \\ =& \boldsymbol{\xi}^c\bar{\boldsymbol{\nabla}}_c T^{a_1\ldots a_n}{}_{b_1\ldots b_m} - T^{c\ldots a_n}{}_{b_1\ldots b_m}\bar{\boldsymbol{\nabla}}_c\boldsymbol{\xi}^{a_1} - \ldots \\ &+ T^{a_1\ldots a_n}{}_{c\ldots b_m}\bar{\boldsymbol{\nabla}}_{b_1}\boldsymbol{\xi}^c + \ldots\ . \end{aligned} \tag{3.47}$$

Hence, we have gauge invariance with respect to infinitesimal UCT since

$$\delta\mathscr{L}_U = \boldsymbol{\xi}^a\partial_a\mathscr{L}_U = \partial_a\left(\boldsymbol{\xi}^a\mathscr{L}_U\right) \tag{3.48}$$

and the action changes by a total derivative. Now, we proceed as before: we compare (3.48) with $\delta\mathscr{L}_U$ caused by an arbitrary variation of $\bar{\mathfrak{g}}^{ab}$, h^{ab} and φ; then we enforce the EOMs and use $\delta\bar{\mathfrak{g}}^{ab}$ for a infinitesimal UCT as given by (3.49).

$$\delta\bar{\mathfrak{g}}^{ab} = \boldsymbol{\xi}^c\bar{\boldsymbol{\nabla}}_c\bar{\mathfrak{g}}^{ab} - 2\bar{\mathfrak{g}}^{c(a}\bar{\boldsymbol{\nabla}}_c\boldsymbol{\xi}^{b)} = -2\bar{\boldsymbol{\nabla}}^{(a}\boldsymbol{\xi}^{b)} \text{ [18]} \tag{3.49}$$

[18] $\Rightarrow \bar{\mathfrak{g}}_{ab}\delta\bar{\mathfrak{g}}^{ab} = 0$

We have

$$\partial_a\left(\boldsymbol{\xi}^a\mathscr{L}_U\right)=\frac{\delta S_U}{\delta h^{ab}}\delta h^{ab}+\frac{\delta S_U}{\delta\varphi}\delta\varphi+\frac{\delta S_U}{\delta\bar{\mathfrak{g}}^{ab}}\delta\bar{\mathfrak{g}}^{ab}+\partial_c\left(\frac{\partial\mathscr{L}_U}{\partial\partial_c h^{ab}}\delta h^{ab}+\frac{\partial\mathscr{L}_U}{\partial\partial_c\varphi}\delta\varphi+\frac{\partial\mathscr{L}_U}{\partial\partial_c\bar{\mathfrak{g}}^{ab}}\delta\bar{\mathfrak{g}}^{ab}\right) \tag{3.50}$$

where $\frac{\delta S_U}{\delta\bar{\mathfrak{g}}^{ab}}\equiv\frac{\partial\mathscr{L}_U}{\partial\bar{\mathfrak{g}}^{ab}}-\partial_c\left(\frac{\partial\mathscr{L}_U}{\partial\partial_c\bar{\mathfrak{g}}^{ab}}\right)$. Using (3.9), (3.23) and integrating this, by parts, one has

$$2\int d^Dx\,\boldsymbol{\xi}^b\bar{\boldsymbol{\nabla}}^a\frac{\delta S_U}{\delta\bar{\mathfrak{g}}^{ab}}+ST=0 \tag{3.51}$$

The transformation parameter is constrained by $\partial_a\boldsymbol{\xi}^a=\bar{\boldsymbol{\nabla}}_a\boldsymbol{\xi}^a=0$ such that we can replace it by the arbitrary parameter $\mathcal{F}^{cb}$: $\boldsymbol{\xi}^b=\partial_c\mathcal{F}^{[cb]}=\bar{\boldsymbol{\nabla}}_c\mathcal{F}^{[cb]}$. After doing this and integrating by parts,

$$-2\int d^Dx\,\mathcal{F}^{[cb]}\bar{\boldsymbol{\nabla}}_c\bar{\boldsymbol{\nabla}}^a\frac{\delta S_U}{\delta\bar{\mathfrak{g}}^{ab}}+ST=0 \tag{3.52}$$

The surface term can be converted into an integral over the boundary through Stokes' Theorem and using an appropriate choice of boundary conditions for the transformation paramete $\mathcal{F}$ it can be made to vanish. Due to arbitrariness of $\mathcal{F}$ one arrives at the following on-shell relation:

$$0=\bar{\boldsymbol{\nabla}}_{[c}\bar{\boldsymbol{\nabla}}^a\frac{\delta S_U}{\delta\bar{\mathfrak{g}}^{b]a}}=\partial_{[c}\bar{\boldsymbol{\nabla}}^a\frac{\delta S_U}{\delta\bar{\mathfrak{g}}^{b]a}} \tag{3.53}$$

Like in section 3.2, this is valid independently of the metric being flat or curved. If we fix inertial coordinates we get the on-shell relation that we were looking for X_{ab} in (2.46): $\partial_{[c}\partial^a\frac{\delta S_U}{\delta\bar{\mathfrak{g}}^{b]a}}\Big|_{\bar{\mathfrak{g}}=\eta}=0$. From now on, we consider the U-covariantizations $S_{U,h}$, $S^{\mathrm{m}}_{U,h}$, $S_{U,\varphi}$ and $S_{U,\mathrm{int}}$ of actions $\mathscr{A}_h$ (invariant under TDiff), $\mathscr{A}^{\mathrm{m}}_h$, $\mathscr{A}_\varphi$ and $\mathscr{A}_{\mathrm{int}}$ of section 2.3, respectively. Hence, instead of (2.46), we have

$$\frac{\delta S^{\mathrm{m}}_{U,\mathrm{int}}}{\delta h^{ab}}=\frac{\delta S_{U,\mathrm{tot}}}{\delta\bar{\mathfrak{g}}^{ab}}+\bar{\boldsymbol{\nabla}}^c\bar{\boldsymbol{\nabla}}^d\Psi_{[c(a][b)d]}[\bar{\mathfrak{g}};h,\varphi]+\bar{\mathfrak{g}}_{ab}\rho[\bar{\mathfrak{g}};h,\varphi]+\bar{\boldsymbol{\nabla}}_a\bar{\boldsymbol{\nabla}}_b\tilde{\rho}[\bar{\mathfrak{g}};h,\varphi] \tag{3.54}$$

$$\Leftrightarrow\bar{\boldsymbol{\nabla}}_{[c}\bar{\boldsymbol{\nabla}}^a\frac{\delta S^{\mathrm{m}}_{U,\mathrm{int}}}{\delta h^{ab}}=\bar{\boldsymbol{\nabla}}_{[c}\bar{\boldsymbol{\nabla}}^a\frac{\delta S_{U,\mathrm{tot}}}{\delta\bar{\mathfrak{g}}^{ab}} \tag{3.55}$$

where $S^{\mathrm{m}}_{U,\mathrm{int}}\equiv\chi^{-1}S^{\mathrm{m}}_{U,h}+S_{U,\mathrm{int}}$ and $S_{U,\mathrm{tot}}\equiv S_{U,h}+S_{U,\varphi}+\chi S^{\mathrm{m}}_{U,\mathrm{int}}$. In the remainder of this section, we use the variations

$$\delta h^{ab}=\boldsymbol{\xi}^c\bar{\boldsymbol{\nabla}}_c h^{ab}-2h^{c(a}\bar{\boldsymbol{\nabla}}_c\boldsymbol{\xi}^{b)} \tag{3.56}$$

and

$$\delta\varphi=\boldsymbol{\xi}^c\bar{\boldsymbol{\nabla}}_c\varphi\;, \tag{3.57}$$

under UCTs (for concreteness, we suppose that φ is a U-scalar) to derive the Gauge/Bianchi Identity associated with these. Integrating (3.50), one obtains

$$\int d^D x \left(\frac{\delta S_U}{\delta h^{ab}} \delta h^{ab} + \frac{\delta S_U}{\delta \varphi} \delta \varphi + \frac{\delta S_U}{\delta \bar{\mathfrak{g}}^{ab}} \delta \bar{\mathfrak{g}}^{ab} \right) + ST = 0 \tag{3.58}$$

Using (3.49), (3.56) and (3.57), on the equation above:

$$\begin{aligned}
&\int d^D x \left(-2 \frac{\delta S_U}{\delta h^{ab}} h^{ac} \bar{\boldsymbol{\nabla}}_c \boldsymbol{\xi}^b + \frac{\delta S_U}{\delta h^{ab}} \boldsymbol{\xi}^c \bar{\boldsymbol{\nabla}}_c h^{ab} + \frac{\delta S_U}{\delta \varphi} \boldsymbol{\xi}^c \bar{\boldsymbol{\nabla}}_c \varphi - 2 \frac{\delta S_U}{\delta \bar{\mathfrak{g}}^{ab}} \bar{\boldsymbol{\nabla}}^a \boldsymbol{\xi}^b \right) + ST \\
&= \int d^D x \left(-2 \frac{\delta S_U}{\delta h^{ab}} h^{ac} \bar{\boldsymbol{\nabla}}_c \boldsymbol{\xi}^b - \boldsymbol{\xi}^j \bar{\boldsymbol{\nabla}}_c T^{ac}_{\text{Can}} \bar{g}_{ab} - 2 \frac{\delta S_U}{\delta \bar{\mathfrak{g}}^{ab}} \bar{\boldsymbol{\nabla}}^a \boldsymbol{\xi}^b \right) + ST = 0
\end{aligned} \tag{3.59}$$

Substituting $\boldsymbol{\xi}^b = \bar{\boldsymbol{\nabla}}_d \mathcal{F}^{[db]}$ and integrating twice by parts, we get

$$-\int d^D x \bar{\boldsymbol{\nabla}}_d \left[2 \bar{\boldsymbol{\nabla}}_c \left(\frac{\delta S_U}{\delta h^{ab}} h^{ac} \right) - \bar{\boldsymbol{\nabla}}_c T^{ac}_{\text{Can}} \bar{\mathfrak{g}}_{ab} + 2 \bar{\boldsymbol{\nabla}}^a \frac{\delta S_U}{\delta \bar{\mathfrak{g}}^{ab}} \right] \mathcal{F}^{[db]} + ST = 0 \tag{3.60}$$

Due to arbitrariness of $\mathcal{F}^{cj}$, this leads to the following off-shell identity that doesn't involve the transformation parameters:

$$\begin{aligned}
&\bar{\boldsymbol{\nabla}}_c \bar{\boldsymbol{\nabla}}_{[d} \left(\bar{\mathfrak{g}}_{b]a} T^{ac}_{\text{Can}} - 2 \frac{\delta S_U}{\delta h^{b]a}} h^{ac} \right) = 2 \bar{\boldsymbol{\nabla}}^a \bar{\boldsymbol{\nabla}}_{[d} \frac{\delta S_U}{\delta \bar{\mathfrak{g}}^{b]a}} = 2 \bar{\boldsymbol{\nabla}}_i \bar{\boldsymbol{\nabla}}_{[d} \left(\frac{\delta S_U}{\delta \bar{\mathfrak{g}}^{b]a}} \bar{\mathfrak{g}}^{ai} \right) \\
&\Rightarrow \bar{\boldsymbol{\nabla}}_c \left(2 \frac{\delta S_U}{\delta \bar{\mathfrak{g}}^{ba}} \bar{\mathfrak{g}}^{ac} - \bar{\mathfrak{g}}_{ba} T^{ac}_{\text{Can}} + 2 \frac{\delta S_U}{\delta h^{ba}} h^{ac} \right) = \bar{\boldsymbol{\nabla}}_b (\rho + \tilde{\rho}) = \partial_b (\rho + \tilde{\rho}) \\
&\Rightarrow 2 \frac{\delta S_U}{\delta \bar{\mathfrak{g}}^{ba}} \bar{\mathfrak{g}}^{ac} - \bar{\mathfrak{g}}_{ba} T^{ac}_{\text{Can}} + 2 \frac{\delta S_U}{\delta h^{ba}} h^{ac} = \bar{\boldsymbol{\nabla}}_b \bar{\boldsymbol{\nabla}}^c \rho' + \delta^c_b \tilde{\rho} + \bar{\boldsymbol{\nabla}}_d \boldsymbol{\psi}_b{}^{[cd]}
\end{aligned} \tag{3.61}$$

where $\bar{\boldsymbol{\nabla}}^b \bar{\boldsymbol{\nabla}}_b \rho' = \rho$. It will turn up useful to write this as

$$-2 \bar{\boldsymbol{\nabla}}_c \bar{\boldsymbol{\nabla}}_{[d} \left(\frac{\delta S_U}{\delta \bar{\mathfrak{g}}^{b]a}} \bar{\mathfrak{g}}^{ac} \right) = \bar{\boldsymbol{\nabla}}_{[d} \left[\left(\bar{\boldsymbol{\nabla}}_{b]} h^{ac} \right) \frac{\delta S_U}{\delta h^{ac}} + \left(\bar{\boldsymbol{\nabla}}_{b]} \varphi \right) \frac{\delta S_U}{\delta \varphi} \right] + 2 \bar{\boldsymbol{\nabla}}_c \bar{\boldsymbol{\nabla}}_{[d} \left(\frac{\delta S_U}{\delta h^{b]a}} h^{ac} \right) \tag{3.62}$$

4 Self-coupling in the Diff case

4.1 Iterative procedure I

Throughout chapter 3, we wrote the functional derivative w.r.t. variables that are constrained:

- $\bar{g}$ is a flat metric ($\mathcal{R}^a{}_{bcd}[\bar{g}] = 0$);
- $\bar{\mathfrak{g}}$, in addition to flatness, is written in unimodular coordinates ($\partial_a|\bar{\mathfrak{g}}| = 0 \Leftrightarrow \mathbf{\Gamma}^b_{ba}[\bar{\mathfrak{g}}] = 0$).

Note that these derivatives must be understood as (γ is an arbitrary uncontrained metric)

$$\frac{\delta S[\bar{g}; h, \varphi]}{\delta \bar{g}} = \lim_{\mathcal{R}^a{}_{bcd}[\gamma] \to 0} \frac{\delta S[\gamma; h, \varphi]}{\delta \gamma} \equiv \lim_{\gamma \to \bar{g}} \frac{\delta S[\gamma; h, \varphi]}{\delta \gamma} \ , \tag{4.1}$$

and the equivalent relation for $\bar{\mathfrak{g}}$, so that we have an uniquely defined functional derivative. Otherwise, under a constrained infinitesimal variation $\delta\bar{g}$ (such that $\mathcal{R}^a{}_{bcd}[\bar{g} + \delta\bar{g}] = 0$), we would have

$$\delta S = \int d^D x \frac{\delta S}{\delta \bar{g}^{ab}} \delta \bar{g}^{ab} + ST \tag{4.2}$$

But if $\delta S + ST = 0$ for any $\delta\bar{g}$ this wouldn't set $\frac{\delta S}{\delta \bar{g}^{ab}}$ to zero, instead there's a myriad of terms that solve this equation. The same happens with the $\bar{\mathfrak{g}}$ case.

It's worth it to delve deeper into this matter. For the next paragraphs we'll use some plain notation. Consider a couple of functional $S^{\mathrm{NM}}[\gamma]$ and $S^{\mathrm{NM}}_U[\gamma]$ that vanish when we impose constraints on γ such that $S^{\mathrm{NM}}[\bar{g}] = 0 = S^{\mathrm{NM}}_U[\bar{\mathfrak{g}}]$ (these are not surface terms). We can vary S^{NM} by varying the metric:

$$S^{\mathrm{NM}}[\gamma + \lambda\delta\gamma] - S^{\mathrm{NM}}[\gamma] \equiv \lambda \delta S^{\mathrm{NM}} \tag{4.3}$$

where λ is a constant that can be set to 1. Note that

$$\lim_{\gamma \to \bar{g}} \lambda \delta S^{\mathrm{NM}} = S^{\mathrm{NM}}[\bar{g} + \lambda\delta\gamma] - S^{\mathrm{NM}}[\bar{g}] = S^{\mathrm{NM}}[\bar{g} + \lambda\delta\gamma] \Rightarrow \lim_{\gamma \to \bar{g}} \lambda\delta S^{\mathrm{NM}} + ST \neq 0 \tag{4.4}$$

By the definition of functional derivative,

$$\lambda \delta S^{\mathrm{NM}} = \int d^D x \, \frac{\delta S^{\mathrm{NM}}}{\delta \gamma} \lambda \delta \gamma + ST + O(\lambda^2) \tag{4.5}$$

Taking into account (4.4),

$$\lim_{\gamma \to \bar{g}} \frac{\delta S^{\mathrm{NM}}}{\delta \gamma} \neq 0 \tag{4.6}$$

However, considering a variation $\delta\gamma = \delta\bar{g}$ (infinitesimal, so that terms $O(\lambda^2)$ can be neglected) such that $\mathcal{R}^a{}_{bcd}[\bar{g}+\delta\bar{g}] = 0$, we have

$$\int d^D x \lim_{\gamma\to\bar{g}} \frac{\delta S^{\rm NM}}{\delta\gamma}\lambda\delta\bar{g} + ST = S^{\rm NM}[\bar{g}+\lambda\delta\bar{g}] - S^{\rm NM}[\bar{g}] = 0 \tag{4.7}$$

Using $S_U^{\rm NM}$ and $\bar{\mathfrak{g}}$, respectively, in place of $S^{\rm NM}$ and $\bar{g}$, we would reach

$$\lim_{\gamma\to\bar{\mathfrak{g}}} \frac{\delta S_U^{\rm NM}}{\delta\gamma} \neq 0 \tag{4.8}$$

and

$$\int d^D x \lim_{\gamma\to\bar{\mathfrak{g}}} \frac{\delta S_U^{\rm NM}}{\delta\gamma}\lambda\delta\bar{\mathfrak{g}} + ST = 0 \tag{4.9}$$

where $\mathcal{R}^a{}_{bcd}[\bar{\mathfrak{g}}+\delta\bar{\mathfrak{g}}] = 0 = \mathbf{\Gamma}^b_{ba}[\bar{\mathfrak{g}}+\delta\bar{\mathfrak{g}}]$. We can conclude that our success in avoiding ambiguities in functional differentiation through (4.1), can be traced back to the use of what's called *minimal coupling*, i.e. we simply replaced the constrained metric by an arbitrary one in S. Have we considered adding terms to S that would vanish in the limit where γ is equally constrained, like $S^{\rm NM}$ (*non-minimal coupling*, that's where "NM" comes from), and we would recover the same form of ambiguity (compare (4.2) and (4.7)/(4.9)) since even though $S^{\rm NM}[\bar{g}]$ vanishes its derivative (4.6)/(4.8) does not. This is completely harmeless anyway since, as you can see in the previous chapter, functional derivatives always arise multiplied by the variation of the respective variable. However, we can and will use non-minimal couplings as a bookkeeping device for identically divergenceless terms and terms whose divergence is a gradient in equations (2.44) and (2.46).

From previous sections, we already know what $\delta\bar{g}$ and $\delta\bar{\mathfrak{g}}$ look like:

$$\delta\bar{g}^{ab} = -2\bar{\nabla}^{(a}\xi^{b)} \tag{3.23}$$

$$\delta\bar{\mathfrak{g}}^{ab} = -2\bar{\boldsymbol{\nabla}}^{(a}\boldsymbol{\xi}^{b)} \tag{3.49}$$

where ξ^b is arbitrary but $\partial_b\boldsymbol{\xi}^b = \bar{\boldsymbol{\nabla}}_b\boldsymbol{\xi}^b = 0$ such that we can replace $\boldsymbol{\xi}^b$ it by the arbitrary parameter $\mathcal{F}^{cb}$: $\boldsymbol{\xi}^b = \partial_c\mathcal{F}^{[cb]} = \bar{\boldsymbol{\nabla}}_c\mathcal{F}^{[cb]}$. Substituting these into (4.7) and (4.9), we have

$$2\int d^D x\, \xi^b\bar{\nabla}^a\left(\lim_{\gamma\to\bar{g}} \frac{\delta S^{\rm NM}}{\delta\gamma^{ab}}\right) + ST = 0 \tag{4.10}$$

and

$$-2\int d^D x\, \mathcal{F}^{[cb]}\bar{\boldsymbol{\nabla}}_c\bar{\boldsymbol{\nabla}}^a\left(\lim_{\gamma\to\bar{\mathfrak{g}}} \frac{\delta S_U^{\rm NM}}{\delta\gamma^{ab}}\right) + ST = 0 \tag{4.11}$$

leading, respectively, to the following of-shell identities:

$$\bar{\nabla}^a \lim_{\gamma\to\bar{g}} \frac{\delta S^{\rm NM}}{\delta\gamma^{ab}} = 0 \tag{4.12}$$

$$\bar{\boldsymbol{\nabla}}_{[c}\bar{\boldsymbol{\nabla}}^a \lim_{\gamma\to\bar{\mathfrak{g}}} \frac{\delta S_U^{\rm NM}}{\delta\gamma^{b]a}} = 0 \tag{4.13}$$

Therefore we can write (3.27) and (3.54), respectively, as

$$\frac{\delta S^{\rm m}_{\rm int}}{\delta h^{ab}} = \left.\frac{\delta S_{\rm tot}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}} \Leftrightarrow \bar{\nabla}^a\frac{\delta S^{\rm m}_{\rm int}}{\delta h^{ab}} = \bar{\nabla}^a\frac{\delta S_{\rm tot}}{\delta\bar{g}^{ab}} \tag{4.14}$$

$$\frac{\delta S^{\rm m}_{U,\rm int}}{\delta h^{ab}} = \left.\frac{\delta S_{U,\rm tot}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}} \Leftrightarrow \bar{\boldsymbol{\nabla}}_{[c}\bar{\boldsymbol{\nabla}}^a\frac{\delta S^{\rm m}_{U,\rm int}}{\delta h^{b]a}} = \bar{\boldsymbol{\nabla}}_{[c}\bar{\boldsymbol{\nabla}}^a\frac{\delta S_{U,\rm tot}}{\delta\bar{\mathfrak{g}}^{b]a}} \tag{4.15}$$

(hence the bookkeeping device) where the vertical bar is our way to point out that minimal coupling is not assumed. We now expand $S^{\rm m}_{\rm int}/S^{\rm m}_{U,\rm int}$ and $S_{\rm tot}/S_{U,\rm tot}$ on the equations above:

$$\frac{\delta S_{\rm int}}{\delta h^{ab}} = -\chi^{-1}\frac{\delta S^{\rm m}_h}{\delta h^{ab}} + \left.\frac{\delta S_h}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}} + \left.\frac{\delta S_\varphi}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}} + \left.\frac{\delta S^{\rm m}_h}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}} + \chi\left.\frac{\delta S_{\rm int}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}} \tag{4.16}$$

$$\frac{\delta S_{U,\rm int}}{\delta h^{ab}} = -\chi^{-1}\frac{\delta S^{\rm m}_{U,h}}{\delta h^{ab}} + \left.\frac{\delta S_{U,h}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}} + \left.\frac{\delta S_{U,\varphi}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}} + \left.\frac{\delta S^{\rm m}_{U,h}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}} + \chi\left.\frac{\delta S_{U,\rm int}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}} \tag{4.17}$$

We can try to find $S_{\rm int}$ and $S_{U,\rm int}$ by writing them as a perturbative series in χ. For example,

$$S_{\rm int} = \chi^{-1}S^{(-1)}_{\rm int} + S^{(0)}_{\rm int} + \chi S^{(1)}_{\rm int} + \chi^2 S^{(2)}_{\rm int} + \ldots \tag{4.18}$$

This way we can solve equations (4.16) and (4.17) iteratively, as we do in (4.19) and (4.20), respectively. Before, let us see that the interacting theory generated this way has no mass terms. Note that, substituting (4.18) in (4.16), implies

$$\begin{cases} \frac{\delta S^{(-1)}_{\rm int}}{\delta h^{ab}} = -\frac{\delta S^{\rm m}_h}{\delta h^{ab}} \Rightarrow S^{(-1)}_{\rm int} = -S^{\rm m}_h + ST \\ \frac{\delta S^{(0)}_{\rm int}}{\delta h^{ab}} = \left.\frac{\delta S_h}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}} + \left.\frac{\delta S_\varphi}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}} + \left.\frac{\delta S^{\rm m}_h}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}} + \left.\frac{\delta S^{(-1)}_{\rm int}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}} \end{cases}$$

$$\Rightarrow \frac{\delta S^{(0)}_{\rm int}}{\delta h^{ab}} = \left.\frac{\delta S_h}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}} + \left.\frac{\delta S_\varphi}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}}$$

The same happens with (4.17). Hence, one could have ignored the mass term from the beggining and written (4.18) without $S^{(-1)}_{\rm int}$, being then led to:

$$\frac{\delta S^{(0)}_{\rm int}}{\delta h^{ab}} = \left.\frac{\delta S_h}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}} + \left.\frac{\delta S_\varphi}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}} \quad \text{and} \quad \frac{\delta S^{(n+1)}_{\rm int}}{\delta h^{ab}} = \left.\frac{\delta S^{(n)}_{\rm int}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}} \tag{4.19}$$

$$\frac{\delta S^{(0)}_{U,\rm int}}{\delta h^{ab}} = \left.\frac{\delta S_{U,h}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}} + \left.\frac{\delta S_{U,\varphi}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}} \quad \text{and} \quad \frac{\delta S^{(n+1)}_{U,\rm int}}{\delta h^{ab}} = \left.\frac{\delta S^{(n)}_{U,\rm int}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}} \tag{4.20}$$

where $n \geq 0$. The iterations last indefinitely (hopefully converging) or until $\frac{\delta S^{(n)}_{\text{int}}}{\delta\gamma^{ab}}\Big|_{\gamma\to\bar{g}} \Big/ \frac{\delta S^{(n)}_{U,\text{int}}}{\delta\gamma^{ab}}\Big|_{\gamma\to\bar{g}}$ vanish.

Let us briefly make mention of the widely used interpretation/motivation for this kind of iterative procedure. Consider the non-interacting theory given by the action $S_h + S_\varphi$ such that the EOM for the field h is $\frac{\delta S_h}{h^{ab}} = 0$. If we couple h to its own EMT and to the EMT of φ then the EOM should be $\frac{\delta S_h}{h^{ab}} = \chi' \frac{\delta S_h}{\delta\gamma^{ab}}|_{\gamma\to\bar{g}} + \chi' \frac{\delta S_\varphi}{\delta\gamma^{ab}}|_{\gamma\to\bar{g}}$. This EOM can be derived from the action $S_h + S_\varphi - \chi' S^{(0)}_{\text{int}}$ if $\frac{\delta S^{(0)}_{\text{int}}}{\delta h^{ab}} = \frac{\delta S_h}{\delta\gamma^{ab}}|_{\gamma\to\bar{g}} + \frac{\delta S_\varphi}{\delta\gamma^{ab}}|_{\gamma\to\bar{g}}$. However, since this new term will also contribute to the EMT, we should add a further term $S^{(1)}_{\text{int}}$ to the action such that $\frac{\delta S^{(1)}_{\text{int}}}{\delta h^{ab}} = \frac{\delta S^{(0)}_{\text{int}}}{\delta\gamma^{ab}}|_{\gamma\to\bar{g}}$. This goes on and on, replicating iterations (4.19).

Taking into account the linearity of (functional) differentiation, one can write $S^{(n)}_{\text{int}} = S^{(n)}_{\text{int},h} + S^{(n)}_{\text{int},h}$ and divide the iterations of (4.19) into two separate sets by:

$$\frac{\delta S^{(0)}_{\text{int},h}}{\delta h^{ab}} = \frac{\delta S_h}{\delta\gamma^{ab}}\bigg|_{\gamma\to\bar{g}} \quad \text{and} \quad \frac{\delta S^{(n+1)}_{\text{int},h}}{\delta h^{ab}} = \frac{\delta S^{(n)}_{\text{int},h}}{\delta\gamma^{ab}}\bigg|_{\gamma\to\bar{g}} \tag{4.21}$$

$$\frac{\delta S^{(0)}_{\text{int},\varphi}}{\delta h^{ab}} = \frac{\delta S_\varphi}{\delta\gamma^{ab}}\bigg|_{\gamma\to\bar{g}} \quad \text{and} \quad \frac{\delta S^{(n+1)}_{\text{int},\varphi}}{\delta h^{ab}} = \frac{\delta S^{(n)}_{\text{int},\varphi}}{\delta\gamma^{ab}}\bigg|_{\gamma\to\bar{g}} \tag{4.22}$$

One can do the same for (4.20). While $S^{(n)}_{\text{int},\varphi}$ always depends on φ, every $S^{(n)}_{\text{int},h}$ is independent of φ. Therefore, it's impossible for any cancelation to occur between terms coming from these and we're not loosing any solution with this division. We'll call equations like (4.21) "self-coupling condition". These will be the centre of our work from now on (while coupling to matter will be neglected).

4.2 Iterative procedure II

It's time to apply the procedure just derived. For the moment let us tackle the Diff case, due to the simplicity of its iterations when compared with TDiff's, such that we'll focus on (4.21). We then start with

$$\mathscr{A}_h = \frac{-1}{2} \int d^D x \, K_{ab}{}^{c}{}_{ef}{}^{d}[\eta] \, \partial_c h^{ab} \partial_d h^{ef} \tag{4.23}$$

where

$$2K_{ab}{}^{c}{}_{ef}{}^{d}[\eta] = \eta^{cd}\eta_{a(e}\eta_{f)b} - b\eta^{cd}\eta_{ab}\eta_{ef} - 2\delta^{c}_{(e}\eta_{f)(a}\delta^{d}_{b)} + a\delta^{c}_{(e}\delta^{d}_{f)}\eta_{ab} + a\delta^{c}_{(a}\delta^{d}_{b)}\eta_{ef} \tag{4.24}$$

with $a = 1 = b$ even though, for completeness, we'll keep these parameters undefined until later in this section. Covariantization of (4.23) gives

$$S_h = \frac{-1}{2} \int d^D x \sqrt{-|\bar{g}|}\, \bar{\nabla}_c h^{ab} \bar{\nabla}_d h^{ef} K_{ab}{}^{c}{}_{ef}{}^{d}[\bar{g}] \tag{4.25}$$

Looking back at (4.23), one sees that only the part of K that is symmetric under $(cab) \leftrightarrow (def)$, $a \leftrightarrow b$ and $e \leftrightarrow f$ contributes to the action[19]. Let us rewrite (4.21) denoting $S^{(N)}_{\text{int},h}$ by S_{N+1} $(N \geq 0)$:

$$\frac{\delta S_1}{\delta h^{ab}} = \frac{\delta S_h}{\delta \gamma^{ab}}\bigg|_{\gamma \to \bar{g}} \quad \text{and} \quad \frac{\delta S_{n+1}}{\delta h^{ab}} = \frac{\delta S_n}{\delta \gamma^{ab}}\bigg|_{\gamma \to \bar{g}} \tag{4.26}$$

where $n \geq 1$. Note that $\frac{\delta S_{h/n}}{\delta \gamma^{ab}}\Big|_{\gamma \to \bar{g}}$, where $S_{h/n} = S_{h/n}[\bar{g}; h]$, stands for

$$\lim_{\gamma \to \bar{g}} \frac{\delta S_{h/n}[\gamma; h]}{\delta \gamma} + \lim_{\gamma \to \bar{g}} \frac{\delta S^{\text{NM}}_{0/n}[\gamma; h]}{\delta \gamma} \equiv \lim_{\gamma \to \bar{g}} \frac{\delta \tilde{S}_{0/n}}{\delta \gamma} \tag{4.27}$$

(recall the beggining of section 4.1). We'll start by assuming minimal coupling such that the second term (a priori undefined) is ignored. In this case, we find convenient not to bring γ to $\bar{g}$ after functional differentiating w.r.t. γ in each step of the iterative procedure and we'll only do it at the end (we'll deal with the consequences of this when concluding the dissertation). We then write

$$\tilde{S}_0 = \frac{-1}{2} \int d^D x \sqrt{-|\gamma|} \nabla_c h^{ab} \nabla_d h^{ef} K_{ab}{}^{c}{}_{ef}{}^{d} \tag{4.28}$$

Throughout this section, if nothing is said, a capital latin letter is to be understood as a sum of products of γ^{ab} and γ_{ab} such that its indices indicate the position (up/down) of non contracted indices in the terms of the sum. If dependence on other metric is made explicit, the same applies with the components of that metric and its inverse. Also in this section we're going to use greek letters for some dummy indices.

Recall that K obeys $K_{ab}{}^{c}{}_{ef}{}^{d} = K_{ef}{}^{d}{}_{ab}{}^{c}$.

$$\frac{\delta \tilde{S}_0}{\delta \gamma^{pq}} = \frac{-1}{2} \sqrt{-|\gamma|} A_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq} \nabla_c h^{ab} \nabla_d h^{ef} - \sqrt{-|\gamma|} B_{pq}{}^{c}{}_{ef}{}^{d}{}_{ab} \nabla_c (h^{ab} \nabla_d h^{ef}) \tag{4.29}$$

A and B are given by the following expressions.

$$A_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq} = \frac{\partial K_{ab}{}^{c}{}_{ef}{}^{d}}{\partial \gamma^{pq}} - \frac{1}{2} \gamma_{pq} K_{ab}{}^{c}{}_{ef}{}^{d} = \frac{1}{\sqrt{-|\gamma|}} \frac{\partial \sqrt{-|\gamma|} K_{ab}{}^{c}{}_{ef}{}^{d}}{\partial \gamma^{pq}} \tag{4.30}$$

$$B_{pq}{}^{c}{}_{ef}{}^{d}{}_{ab} = \gamma_{p\nu} \gamma_{q\rho} \gamma^{\upsilon\tau} \Delta^{c\nu\rho}{}_{\mu\tau(a} K_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d} \equiv \tilde{\Delta}^{c}{}_{pq\mu}{}^{\upsilon}{}_{(a} K_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d} \tag{4.31}$$

[19]Note also that exchanging c and d would only contribute with a surface term.

where $\Delta^{c\nu\rho}{}_{\mu\tau a} \equiv \left[\delta^c_\mu \delta^{(\nu}_a \delta^{\rho)}_\tau + \delta^c_a \delta^{(\nu}_\mu \delta^{\rho)}_\tau - \delta^c_\tau \delta^{(\rho}_\mu \delta^{\nu)}_a\right]$. Thus we must have

$$\tilde{S}_1 = \int d^D x \sqrt{-|\gamma|} X_{ab}{}^c{}_{ef}{}^d{}_{pq} h^{pq} \nabla_c h^{ab} \nabla_d h^{ef} \tag{4.32}$$

since the self-coupling condition implies that $\tilde{S}_1$ contains three h and two ∇ and any action term involving these will equal (4.32) up to ST.

$$\frac{\delta \tilde{S}_1}{\delta h^{pq}} = \sqrt{-|\gamma|} X_{ab}{}^c{}_{ef}{}^d{}_{pq} \nabla_c h^{ab} \nabla_d h^{ef} - 2\sqrt{-|\gamma|} X_{pq}{}^c{}_{ef}{}^d{}_{ab} \nabla_c (h^{ab} \nabla_d h^{ef}) \tag{4.33}$$

Comparing (4.29) and (4.33) leads to

$$\frac{\delta \tilde{S}_1}{\delta h^{pq}} = \frac{\delta \tilde{S}_0}{\delta \gamma^{pq}} \Rightarrow \begin{cases} X_{ab}{}^c{}_{ef}{}^d{}_{pq} = \frac{-1}{2} A_{ab}{}^c{}_{ef}{}^d{}_{pq} \\ 2X_{pq}{}^c{}_{ef}{}^d{}_{ab} = B_{pq}{}^c{}_{ef}{}^d{}_{ab} \end{cases} \Leftrightarrow \begin{cases} X_{ab}{}^c{}_{ef}{}^d{}_{pq} = \frac{-1}{2} A_{ab}{}^c{}_{ef}{}^d{}_{pq} \\ B_{pq}{}^c{}_{ef}{}^d{}_{ab} + A_{pq}{}^c{}_{ef}{}^d{}_{ab} = 0 \end{cases} \tag{4.34}$$

Equations like $B + A = 0$ above will appear in each step of the iterations. We call them "consistency requirements". For the moment, we assure the consistency requirement in the first step is verified by assuming that K obeys

$$\sqrt{-|\gamma|} \tilde{\Delta}^c{}_{pq\mu}{}^\upsilon{}_{(a} K_{b)\upsilon}{}^\mu{}_{ef}{}^d + \frac{\partial \sqrt{-|\gamma|} K_{pq}{}^c{}_{ef}{}^d}{\partial \gamma^{ab}} = 0 \tag{4.35}$$

(This comes from substituting the expressions for A and B in the consistency requirement). Taking (4.34) together with (4.32) into account, we start the second step of the iterative procedure with

$$\tilde{S}_1 = \frac{-1}{2} \int d^D x \sqrt{-|\gamma|} A_{ab}{}^c{}_{ef}{}^d{}_{ij} h^{ij} \nabla_c h^{ab} \nabla_d h^{ef} \tag{4.36}$$

and we find that

$$\frac{\delta \tilde{S}_1}{\delta \gamma^{pq}} = \frac{-1}{2} \sqrt{-|\gamma|} C_{ab}{}^c{}_{ef}{}^d{}_{ijpq} h^{ij} \nabla_c h^{ab} \nabla_d h^{ef} - \sqrt{-|\gamma|} D_{pq}{}^c{}_{ef}{}^d{}_{ijab} \nabla_c (h^{ij} h^{ab} \nabla_d h^{ef}) \tag{4.37}$$

where C and D are given by the following expressions.

$$C_{ab}{}^c{}_{ef}{}^d{}_{ijpq} = \frac{1}{\sqrt{-|\gamma|}} \frac{\partial \sqrt{-|\gamma|} A_{ab}{}^c{}_{ef}{}^d{}_{ij}}{\partial \gamma^{pq}} = \frac{1}{\sqrt{-|\gamma|}} \frac{\partial^2 \sqrt{-|\gamma|} K_{ab}{}^c{}_{ef}{}^d}{\partial \gamma^{pq} \partial \gamma^{ij}} \tag{4.38}$$

$$\begin{aligned} D_{pq}{}^c{}_{ef}{}^d{}_{ijab} &= \frac{1}{2} \tilde{\Delta}^c{}_{pq\mu}{}^\upsilon{}_{(a} A_{b)\upsilon}{}^\mu{}_{ef}{}^d{}_{ij} + (ab) \leftrightarrow (ij) \\ &= \frac{1}{2\sqrt{-|\gamma|}} \tilde{\Delta}^c{}_{pq\mu}{}^\upsilon{}_{(a} \frac{\partial \sqrt{-|\gamma|} K_{b)\upsilon}{}^\mu{}_{ef}{}^d}{\partial \gamma^{ij}} + (ab) \leftrightarrow (ij) \end{aligned} \tag{4.39}$$

(Hence, $C_{\dots ijpq} = C_{\dots pqij}$). Any action term involving four h and two ∇ will equal, up to ST,

$$\tilde{S}_2 = \int d^D x \sqrt{-|\gamma|} Y_{ab}{}^{c}{}_{ef}{}^{d}{}_{ijpq} h^{pq} h^{ij} \nabla_c h^{ab} \nabla_d h^{ef} \tag{4.40}$$

and we obtain an expression similar to $\frac{\delta \tilde{S}_1}{h^{pq}}$ with the difference that the "$\nabla h \nabla h$" term is multiplied by 2 (corresponding to the two h's outside ∇):

$$\frac{\delta \tilde{S}_2}{\delta h^{pq}} = 2\sqrt{-|\gamma|} Y_{ab}{}^{c}{}_{ef}{}^{d}{}_{ijpq} h^{ij} \nabla_c h^{ab} \nabla_d h^{ef} - 2\sqrt{-|\gamma|} Y_{pq}{}^{c}{}_{ef}{}^{d}{}_{ijab} \nabla_c (h^{ij} h^{ab} \nabla_d h^{ef}) \tag{4.41}$$

$$\frac{\delta \tilde{S}_2}{\delta h^{pq}} = \frac{\delta \tilde{S}_1}{\delta \gamma^{pq}} \Rightarrow \begin{cases} 2Y_{ab}{}^{c}{}_{ef}{}^{d}{}_{ijpq} = \frac{-1}{2} C_{ab}{}^{c}{}_{ef}{}^{d}{}_{ijpq} \\ 2Y_{pq}{}^{c}{}_{ef}{}^{d}{}_{ijab} = D_{pq}{}^{c}{}_{ef}{}^{d}{}_{ijab} \end{cases} \tag{4.42}$$

Like before, substituting the top equation into the bottom one entails the consistency requirement: $2D_{pq}{}^{c}{}_{ef}{}^{d}{}_{ijab} + C_{pq}{}^{c}{}_{ef}{}^{d}{}_{ijab} = 0$; which is equivalent to

$$\tilde{\Delta}^{c}{}_{pq\mu}{}^{\upsilon}{}_{(a} \frac{\partial \sqrt{-|\gamma|} K_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d}}{\partial \gamma^{ij}} + (ab) \leftrightarrow (ij) + \frac{\partial^2 \sqrt{-|\gamma|} K_{pq}{}^{c}{}_{ef}{}^{d}}{\partial \gamma^{ab} \partial \gamma^{ij}} = 0 \tag{4.43}$$

If we follow

> Procedure A: 1) differentiate both sides of assumption (4.35) with respect to γ^{ij}; 2) take into account the algebraic relation $\frac{\partial \tilde{\Delta}^{c}{}_{pq\mu}{}^{\upsilon}{}_{a}}{\partial \gamma^{ij}} = -\tilde{\Delta}^{c}{}_{pq\theta}{}^{\gamma}{}_{(i} \tilde{\Delta}^{\theta}{}_{\gamma j)\mu}{}^{\upsilon}{}_{a}$; 3) substitute (4.35) to get rid of the term with two $\tilde{\Delta}$'s.

we conclude that satisfying (4.43) is actually assured by our previous assumption. We now easily see that starting with ($n \geq 0$)

$$\tilde{S}_n = \frac{-1}{2n!} \int d^D x \frac{\partial^n \sqrt{-|\gamma|} K_{ab}{}^{c}{}_{ef}{}^{d}}{\partial \gamma^{kl} \dots \partial \gamma^{ij}} \overbrace{h^{ij} \dots h^{kl}}^{n \text{ times}} \nabla_c h^{ab} \nabla_d h^{ef} \tag{4.44}$$

such that $\tilde{S}_0$, $\tilde{S}_1$ and $\tilde{S}_2$ match the previously obtained expressions, we obtain

$$\begin{aligned} \frac{\delta \tilde{S}_n}{\delta \gamma^{pq}} &= \frac{-1}{2n!} \frac{\partial^{n+1} \sqrt{-|\gamma|} K_{ab}{}^{c}{}_{ef}{}^{d}}{\partial \gamma^{pq} \partial \gamma^{kl} \dots \partial \gamma^{ij}} h^{ij} \dots h^{kl} \nabla_c h^{ab} \nabla_d h^{ef} \\ &- \frac{1}{n!\,(n+1)} \left[\tilde{\Delta}^{c}{}_{pq\mu}{}^{\upsilon}{}_{(a} \frac{\partial^n \sqrt{-|\gamma|} K_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d}}{\partial \gamma^{kl} \dots \partial \gamma^{ij}} + (ab) \leftrightarrow (ij) + \dots + (ab) \leftrightarrow (kl) \right] \nabla_c (h^{ij} \dots h^{kl} h^{ab} \nabla_d h^{ef}) \end{aligned} \tag{4.45}$$

Our ansatz to the self-coupling condition is

$$\tilde{S}^*_{n+1} = \int d^D x \sqrt{-|\gamma|}\, Z_{ab}{}^{c}{}_{ef}{}^{d}{}_{ij\dots klpq} h^{pq} h^{ij} \dots h^{kl} \nabla_c h^{ab} \nabla_d h^{ef} \tag{4.46}$$

such that

$$\frac{\delta\tilde{S}^*_{n+1}}{\delta h^{pq}} = \sqrt{-|\gamma|}\left[(n+1)Z_{ab}{}^{c}{}_{ef}{}^{d}{}_{ij...klpq}h^{ij}...\,h^{kl}\nabla_c h^{ab}\nabla_d h^{ef} - 2\,Z_{pq}{}^{c}{}_{ef}{}^{d}{}_{ij...klab}\nabla_c(h^{ij}...\,h^{kl}h^{ab}\nabla_d h^{ef})\right] \tag{4.47}$$

Comparing this with (4.45), we obtain

$$(n+1)\sqrt{-|\gamma|}\,Z_{ab}{}^{c}{}_{ef}{}^{d}{}_{ij...klpq} = \frac{-1}{2n!}\frac{\partial^{n+1}\sqrt{-|\gamma|}K_{ab}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{pq}\partial\gamma^{kl}...\,\partial\gamma^{ij}} \qquad \text{and} \tag{4.48}$$

$$2\sqrt{-|\gamma|}\,Z_{pq}{}^{c}{}_{ef}{}^{d}{}_{ij...klab} = \frac{1}{(n+1)!}\left[\tilde{\Delta}^{c}{}_{pq\mu}{}^{\upsilon}{}_{(a}\frac{\partial^{n}\sqrt{-|\gamma|}K_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d}}{\partial\gamma^{kl}...\,\partial\gamma^{ij}} + (ab)\leftrightarrow(ij) + ... + (ab)\leftrightarrow(kl)\right] \tag{4.49}$$

Equation (4.48) leads to $\tilde{S}^*_{n+1} = \tilde{S}_{n+1}$ as given by (4.44) and once more we get a consistency requirement:

$$\tilde{\Delta}^{c}{}_{pq\mu}{}^{\upsilon}{}_{(a}\frac{\partial^{n}\sqrt{-|\gamma|}K_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d}}{\partial\gamma^{kl}...\,\partial\gamma^{ij}} + (ab)\leftrightarrow(ij) + ... + (ab)\leftrightarrow(kl) + \frac{\partial^{n+1}\sqrt{-|\gamma|}K_{pq}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{ab}\partial\gamma^{kl}...\,\partial\gamma^{ij}} = 0 \tag{4.50}$$

Just like we used Procedure A to show that, written with K explicit, the consistency requirement in step 1 of the iterations is enough to assure the consistency requirement in step 2, we can do it from step 2 to step 3, from step 3 to step 4, etc. Therefore satisfying all the requirements depends uniquely on assumption (4.35). This is due to the form of equation (4.50) and relies heavily on the algebraic relation "$\frac{\partial\tilde{\Delta}}{\partial\gamma} = -\tilde{\Delta}^2$", as pointed out in [10] where this relation is rigorously shown. Note that, since

$$\frac{\partial^{n}\sqrt{-|\gamma|}K_{ab}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{kl}...\,\partial\gamma^{ij}}(x) = \int d^Dx_1...\,d^Dx_n\frac{\delta^{n}\sqrt{-|\gamma|}K_{ab}{}^{c}{}_{ef}{}^{d}(x)}{\delta\gamma^{kl}(x_1)...\,\delta\gamma^{ij}(x_n)}\;, \tag{4.51}$$

we have

$$\tilde{S}_n = \frac{-1}{2n!}\int d^Dx\,d^Dx_1...\,d^Dx_n\frac{\delta^{n}\sqrt{|\breve{g}|}K_{ab}{}^{c}{}_{ef}{}^{d}[\breve{g}](x)}{\delta\breve{g}^{kl}(x_1)...\,\delta\breve{g}^{ij}(x_n)}\bigg|_{\breve{g}\to\gamma}h^{ij}(x_1)...\,h^{kl}(x_n)\nabla_c h^{ab}(x)\nabla_d h^{ef}(x) \tag{4.52}$$

such that

$$\begin{aligned}\sum_{n=0}^{\infty}\chi^n\tilde{S}_n &= \frac{-1}{2}\int d^Dx\,\left(\sqrt{|\breve{g}|}K_{ab}{}^{c}{}_{ef}{}^{d}[\breve{g}]\right)[\gamma^{ab}+\chi h^{ab}]\nabla_c h^{ab}\nabla_d h^{ef}\\ &= \frac{-1}{2\chi^2}\int d^Dx\,\sqrt{-|g|}K_{ab}{}^{c}{}_{ef}{}^{d}[g]\nabla_c g^{ab}\nabla_d g^{ef}\end{aligned} \tag{4.53}$$

where $g^{ab} = \gamma^{ab} + \chi h^{ab}$. To conclude the iterative procedure note that

$$\lim_{\gamma\to\bar{g}} \sum_{n=0}^{\infty} \chi^n \tilde{S}_n = \sum_{n=0}^{\infty} \chi^n S_n = S_h + \sum_{n=0}^{\infty} \chi^{n+1} S^{(n)}_{\text{int},h} \equiv S_h + \chi S_{\text{int},h} \tag{4.54}$$

The following claim is crucial.

Claim A: (4.54), with $a = 1 = b$, equals the Einstein-Hilbert action (and this determines the value of the coupling constant χ).

Before demonstrating this, we'll end this section by addressing the fact that assumption (4.35) is not true for parameters $a = 1 = b$. This problem is solved through non-minimal coupling.

Recall that $\gamma \to \bar{g}$ is an abbreviation for $\mathcal{R}^a{}_{bcd}[\gamma] \to 0$. Since $\mathcal{R}^a{}_{bcd}$ is quadratic in ∇, we must use non-minimal coupling terms of the form

$$S_0^{\text{NM}} = \frac{1}{2}\int d^Dx\sqrt{-|\gamma|}Q_i{}^{bcj}{}_{afed}\mathcal{R}^i{}_{bcj}h^{af}h^{ed} \tag{4.55}$$

(where $Q_i{}^{bcj}{}_{afed} = Q_i{}^{bcj}{}_{edaf}$). We use the fact that

$$\mathcal{R}^i{}_{bcj}[\gamma] = \mathcal{R}^i{}_{bcj}[\gamma+\delta\gamma] - 2\nabla_{[c}\delta\mathcal{C}^i_{j]b} - 2\delta\mathcal{C}^i_{d[c}\delta\mathcal{C}^d_{j]b} \tag{4.56}$$

where $\delta\mathcal{C}^i_{jb} = \frac{1}{2}(\gamma^{iv} + \delta\gamma^{iv})\nabla_\mu(\gamma_{\nu\rho} + \delta\gamma_{\nu\rho})\Delta^{\mu\nu\rho}{}_{jvb}$ to see how the Riemann changes with variations of γ:

$$\delta\mathcal{R}^i{}_{bcj} = 2\nabla_{[c}\delta\mathcal{C}^i_{j]b} = -\tilde{\Delta}^\mu{}_{\theta\phi b}{}^i{}_{[j}\nabla_{c]}\nabla_\mu\delta\gamma^{\theta\phi} \tag{4.57}$$

where we have neglected terms of quadratic order in $\delta\gamma$. Hence, we have

$$\begin{aligned}\frac{\delta S_0^{\text{NM}}}{\delta\gamma^{pq}} &= \frac{1}{2}\frac{\partial\sqrt{-|\gamma|}Q_i{}^{bcj}{}_{afed}}{\partial\gamma^{pq}}\mathcal{R}^i{}_{bcj}h^{af}h^{ed} - \frac{\sqrt{-|\gamma|}}{2}Q_i{}^{\tau dj}{}_{abef}\tilde{\Delta}^c{}_{pq\tau}{}^i{}_{[j}\nabla_c\nabla_{d]}h^{ab}h^{ef} \\ &= \frac{1}{2}\frac{\partial\sqrt{-|\gamma|}Q_i{}^{bcj}{}_{afed}}{\partial\gamma^{pq}}\mathcal{R}^i{}_{bcj}h^{af}h^{ed} - \sqrt{-|\gamma|}Q_i{}^{\tau[dj]}{}_{abef}\tilde{\Delta}^c{}_{pq\tau}{}^i{}_j\nabla_c(h^{ab}\nabla_d h^{ef})\end{aligned} \tag{4.58}$$

According to (4.27), after differentiating we should bring γ to $\bar{g}$, such that the first term above vanishes. To stay closer to the way we ran the iterations when the minimal coupling was assumed, we'll keep using γ and simply ignore terms proportional to the Riemann. Let us go over some of the previous formulas to see how they change with non-minimal coupling:

$$\tilde{S}_0 = \frac{-1}{2}\int d^Dx\sqrt{-|\gamma|}\nabla_c h^{ab}\nabla_d h^{ef}K_{ab}{}^c{}_{ef}{}^d + S_0^{\text{NM}} \Rightarrow$$

$$\frac{\delta\tilde{S}_0}{\delta\gamma^{pq}} = \frac{-1}{2}\sqrt{-|\gamma|}A_{ab}{}^c{}_{ef}{}^d{}_{pq}\nabla_c h^{ab}\nabla_d h^{ef} - \sqrt{-|\gamma|}\left[B_{pq}{}^c{}_{ef}{}^d{}_{ab} + Q_i{}^{\tau[dj]}{}_{abef}\tilde{\Delta}^c{}_{pq\tau}{}^i{}_j\right]\nabla_c(h^{ab}\nabla_d h^{ef})$$

Then, the $B+A=0$ equation in (4.34) becomes

$$B_{pq\ ef\ ab}^{\ \ c\ \ d}+A_{pq\ ef\ ab}^{\ \ c\ \ d}=-Q_{i\ \ \ \ abef}^{\ \tau[dj]}\tilde{\Delta}^{c}{}_{pq\tau}{}^{i}{}_{j}\Leftrightarrow \tag{4.59}$$

$$\sqrt{-|\gamma|}\tilde{\Delta}^{c}{}_{pq\mu}{}^{\upsilon}{}_{(a}K_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d}+\frac{\partial\sqrt{-|\gamma|}K_{pq}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{ab}}=-\sqrt{-|\gamma|}\tilde{\Delta}^{c}{}_{pq\tau}{}^{i}{}_{j}Q_{i}{}^{\tau[dj]}{}_{abef} \tag{4.60}$$

Using Procedure A on (4.60), we obtain

$$\tilde{\Delta}^{c}{}_{pq\mu}{}^{\upsilon}{}_{(a}\frac{\partial\sqrt{-|\gamma|}K_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d}}{\partial\gamma^{\theta\phi}}+(ab)\leftrightarrow(\theta\phi)+\frac{\partial^{2}\sqrt{-|\gamma|}K_{pq}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{ab}\partial\gamma^{\theta\phi}}=-\tilde{\Delta}^{c}{}_{pq\tau}{}^{i}{}_{j}\frac{\partial\sqrt{-|\gamma|}Q_{i}{}^{\tau[dj]}{}_{abef}}{\partial\gamma^{\theta\phi}} \tag{4.61}$$

This will turn up useful in a moment. Dividing (4.60) by $\sqrt{-|\gamma|}$ we get an equation that is solved by

$$Q_{i}{}^{\tau dj}{}_{abef}=\frac{1}{2}\left(\delta_{i}^{d}\gamma^{\tau j}\left[\gamma_{e(a}\gamma_{b)f}+\frac{1}{2}\gamma_{ab}\gamma_{ef}\right]-\delta^{\tau}_{(a}\delta^{j}_{b)}\delta^{d}_{i}\gamma_{ef}-\delta^{\tau}_{(e}\delta^{j}_{f)}\delta^{d}_{i}\gamma_{ab}\right) \tag{4.62}$$

(see Appendix C) such that S_0^{NM} is completely determined:

$$S_0^{\mathrm{NM}}=\frac{1}{2}\int d^Dx\sqrt{-|\gamma|}\left[\frac{\mathcal{R}}{2}\left(\gamma_{ae}\gamma_{df}+\frac{1}{2}\gamma_{af}\gamma_{ed}\right)-\mathcal{R}_{af}\gamma_{ed}\right]h^{af}h^{ed} \tag{4.63}$$

If we start the second step of the iterative procedure with (4.36), the self-coupling condition will require equation (4.43). However (4.61) tells us that to satisfy this $\tilde{\Delta}^{c}{}_{pq\tau}{}^{i}{}_{j}\frac{\partial\sqrt{-|\gamma|}Q_{i}{}^{\tau[dj]}{}_{abef}}{\partial\gamma^{\theta\phi}}$ should vanish. Since this doesn't happen, $\tilde{S}_1$ must also contain a non-minimal coupling term:

$$S_1^{\mathrm{NM}}=\int d^Dx\sqrt{-|\gamma|}\mathcal{R}^{i}{}_{\tau dj}Q_{i}{}^{\tau dj}{}_{abefst}h^{st}h^{ab}h^{ef} \tag{4.64}$$

where $Q_{i}{}^{\tau dj}{}_{abefst}=Q_{i}{}^{\tau dj}{}_{efabst}=Q_{i}{}^{\tau dj}{}_{stefab}$ and thus we have

$$\begin{aligned}\frac{\delta S_1^{\mathrm{NM}}}{\delta\gamma^{pq}}&=-\sqrt{-|\gamma|}Q_{i}{}^{\tau dj}{}_{abefst}\tilde{\Delta}^{c}{}_{pq\tau}{}^{i}{}_{[j}\nabla_c\nabla_{d]}h^{st}h^{ab}h^{ef}\\&=-3\sqrt{-|\gamma|}Q_{i}{}^{\tau[dj]}{}_{abefst}\tilde{\Delta}^{c}{}_{pq\tau}{}^{i}{}_{j}\nabla_c(h^{st}h^{ab}\nabla_d h^{ef})\end{aligned} \tag{4.65}$$

(already ignoring terms proportional to the Riemann). Hence instead of (4.37) we now have

$$\begin{aligned}\frac{\delta\tilde{S}_1}{\delta\gamma^{pq}}&=\frac{-1}{2}\frac{\partial^{2}\sqrt{-|\gamma|}K_{ab}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{pq}\partial\gamma^{st}}h^{st}\nabla_c h^{ab}\nabla_d h^{ef}\\&-\frac{1}{2}\left[\tilde{\Delta}^{c}{}_{pq\mu}{}^{\upsilon}{}_{(a}\frac{\partial\sqrt{-|\gamma|}K_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d}}{\partial\gamma^{st}}+(ab)\leftrightarrow(st)+6\sqrt{-|\gamma|}Q_{i}{}^{\tau[dj]}{}_{abefst}\tilde{\Delta}^{c}{}_{pq\tau}{}^{i}{}_{j}\right]\nabla_c(h^{st}h^{ab}\nabla_d h^{ef})\end{aligned} \tag{4.66}$$

This time the self-coupling condition requires $Q_i{}^{\tau[dj]}{}_{abefst}$ to solve

$$\tilde{\Delta}^c{}_{pq\mu}{}^{\upsilon}{}_{(a} \frac{\partial\sqrt{-|\gamma|}K_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d}}{\partial\gamma^{st}} + (ab) \leftrightarrow (st) + \frac{\partial^2\sqrt{-|\gamma|}K_{pq}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{ab}\partial\gamma^{st}} = -6\sqrt{-|\gamma|}Q_i{}^{\tau[dj]}{}_{abefst}\tilde{\Delta}^c{}_{pq\tau}{}^{i}{}_{j} \tag{4.67}$$

Taking into account equation (4.61), this reduces to

$$\frac{\partial\sqrt{-|\gamma|}Q_i{}^{\tau[dj]}{}_{abef}}{\partial\gamma^{st}}\tilde{\Delta}^c{}_{pq\tau}{}^{i}{}_{j} = 6\sqrt{-|\gamma|}Q_i{}^{\tau[dj]}{}_{abefst}\tilde{\Delta}^c{}_{pq\tau}{}^{i}{}_{j} \tag{4.68}$$

which can be shown to be equivalent to

$$Q_i{}^{\tau[dj]}{}_{abefst} = \frac{1}{6\sqrt{-|\gamma|}}\frac{\partial\sqrt{-|\gamma|}Q_i{}^{\tau[dj]}{}_{abef}}{\partial\gamma^{st}} \tag{4.69}$$

Let us generalise the procedure above so that every $\tilde{S}_n$ contains a non-minimal coupling term:

$$S_n^{\mathrm{NM}} = \int d^Dx\sqrt{-|\gamma|}\mathcal{R}^i{}_{\tau dj}Q_i{}^{\tau dj}{}_{abefst...kl}\overbrace{h^{kl}...\,h^{st}}^{n \text{ times}}h^{ab}h^{ef} \tag{4.70}$$

$$\Rightarrow \frac{\delta S_n^{\mathrm{NM}}}{\delta\gamma^{pq}} = -(n+2)\sqrt{-|\gamma|}Q_i{}^{\tau[dj]}{}_{abefst...kl}\tilde{\Delta}^c{}_{pq\tau}{}^{i}{}_{j}\nabla_c(h^{kl}...\,h^{st}h^{ab}\nabla_d h^{ef}) \tag{4.71}$$

(already ignoring terms proportional to the Riemann). Hence instead of (4.45) we now have

$$\begin{aligned}\frac{\delta\tilde{S}_n}{\delta\gamma^{pq}} = \frac{-1}{2n!}\frac{\partial^{n+1}\sqrt{-|\gamma|}K_{ab}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{pq}\partial\gamma^{kl}...\,\partial\gamma^{st}}h^{kl}...\,h^{st}\nabla_c h^{ab}\nabla_d h^{ef} - \frac{1}{(n+1)!}\Bigg[\tilde{\Delta}^c{}_{pq\mu}{}^{\upsilon}{}_{(a}\frac{\partial^n\sqrt{-|\gamma|}K_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d}}{\partial\gamma^{kl}...\,\partial\gamma^{st}} \\ + (ab) \leftrightarrow (st) + ... + (ab) \leftrightarrow (kl) + (n+2)!\,\sqrt{-|\gamma|}Q_i{}^{\tau[dj]}{}_{abefst...kl}\tilde{\Delta}^c{}_{pq\tau}{}^{i}{}_{j}\Bigg]\nabla_c(h^{kl}...\,h^{st}h^{ab}\nabla_d h^{ef})\end{aligned} \tag{4.72}$$

This time the self-coupling condition requires $Q_i{}^{\tau[dj]}{}_{abefst...kl}$ to solve

$$\begin{aligned}\tilde{\Delta}^c{}_{pq\mu}{}^{\upsilon}{}_{(a}\frac{\partial^n\sqrt{-|\gamma|}K_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d}}{\partial\gamma^{kl}...\,\partial\gamma^{st}} + (ab) \leftrightarrow (st) + ... + (ab) \leftrightarrow (kl) + \frac{\partial^{n+1}\sqrt{-|\gamma|}K_{pq}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{ab}\partial\gamma^{kl}...\,\partial\gamma^{st}} \\ = -(n+2)!\,\sqrt{-|\gamma|}Q_i{}^{\tau[dj]}{}_{abefst...kl}\tilde{\Delta}^c{}_{pq\tau}{}^{i}{}_{j}\end{aligned} \tag{4.73}$$

By repeatedly applying Procedure A n times starting with (4.60), we get

$$\begin{aligned}Q_i{}^{\tau[dj]}{}_{abefst...kl}(x) &= \frac{(n+2)!^{-1}}{\sqrt{-|\gamma|}}\frac{\partial^n\sqrt{-|\gamma|}Q_i{}^{\tau[dj]}{}_{abef}}{\partial\gamma^{kl}...\,\partial\gamma^{st}}(x) \\ &= \frac{(n+2)!^{-1}}{\sqrt{-|\gamma|}}\int d^Dx_1...\,d^Dx_n\frac{\delta^n\sqrt{-|\gamma|}Q_i{}^{\tau[dj]}{}_{abef}(x)}{\delta\gamma^{kl}(x_1)...\,\delta\gamma^{st}(x_n)}\end{aligned} \tag{4.74}$$

such that

$$S_n^{\rm NM} = \frac{1}{(n+2)!}\int d^D x\, \mathcal{R}^i{}_{\tau dj} h^{ab}h^{ef}\int d^D x_1 ... \, d^D x_n \frac{\delta^n \sqrt{-|\gamma|} Q_i{}^{\tau[dj]}{}_{abef}(x)}{\delta\gamma^{kl}(x_1)...\,\delta\gamma^{st}(x_n)} h^{kl}(x_1)...\, h^{st}(x_n) \tag{4.75}$$

which agrees with the previous expressions for $S_0^{\rm NM}$ and $S_1^{\rm NM}$.

4.3 Bi-connection GR

In this section we back up Claim A. This is only needed because (4.53) and (4.54) happen to differ from the embodiment of Einstein-Hilbert action we're used to. We then resort to Tomboulis' formulation of GR with two connections (this is very much related to Rosen's bimetric formulation [37]).

Consider the covariant derivative $\hat\nabla$ of the Levi-Civita connection $\hat\Gamma[g]$ in terms of the derivative $\check\nabla$ of a torsion-free connection $\check\Gamma$. For an arbitrary weight-ω tensor density $V^{b_1...b_n}{}_{a_1...a_n}$,

$$\begin{aligned}\hat\nabla_c V^{b_0...b_n}{}_{a_0...a_n} = \check\nabla_c V^{b_0...b_n}{}_{a_0...a_n} + \sum_{i=0}^{n}\mathcal{C}^{b_i}_{cd}V^{b_0...d...b_n}{}_{a_0...a_n} - \sum_{i=0}^{n}\mathcal{C}^{d}_{ca_i}V^{b_0...b_n}{}_{a_0...d...a_n} \\ -\omega\mathcal{C}^d_{dc}V^{b_0...b_n}{}_{a_0...a_n}\end{aligned} \tag{4.76}$$

where

$$\mathcal{C}^a_{bc} = \frac{1}{2}g^{ad}\left(\check\nabla_b g_{cd} + \check\nabla_c g_{bd} - \check\nabla_d g_{bc}\right) = \frac{1}{2}g^{ad}\check\nabla_\mu g_{\nu\rho}\Delta^{\mu\nu\rho}{}_{bdc} \tag{4.77}$$

The fact that $\check\nabla_a(g^{bd}g_{dc}) = \check\nabla_a\delta^b_c = 0$ implies the useful equality: $g^{bd}\check\nabla_a g_{dc} = -g_{dc}\check\nabla_a g^{bd}$. This allows us to rewrite $\mathcal{C}$ as

$$\mathcal{C}^a_{bc} = \frac{-1}{2}\left(g_{bd}\check\nabla_c g^{da} + g_{cd}\check\nabla_b g^{da} - g_{be}g_{cf}g^{ad}\check\nabla_d g^{ef}\right) = \frac{-1}{2}g_{\nu\rho}\Delta^{\mu\nu\rho}{}_{bdc}\check\nabla_\mu g^{ad}\,. \tag{4.78}$$

(4.77) is obtained from the condition of metric compatibility, $\hat\nabla g_{ab} = 0$. We could have arrived to (4.78) first from the equivalent condition $\hat\nabla g^{ab} = 0$. If one expands $\hat\nabla$ and $\check\nabla$ in terms of their connections, in (4.76), it becomes clear that $\hat\Gamma[g] = \check\Gamma + \mathcal{C}$. Hence, $\mathcal{C}$ is the difference between two connections, therefore a tensor. One can see this explicitly through (4.77) and (4.78). We now move on to consider the Riemann tensor[20]:

$$\mathcal{R}^d{}_{cab}[\hat\Gamma]v_d = -\left[\hat\nabla_a\,,\hat\nabla_b\right]v_c\,, \tag{4.79}$$

[20]Since $\check\Gamma$ is not a priori associated with any metric, we write $\mathcal{R}[\hat\Gamma]$ intead $\mathcal{R}[g]$ in this section.

where v is an arbitrary covector. Using (4.76) in (4.79) one can write the Riemann in terms of $\check{\Gamma}$:

$$\mathcal{R}^d{}_{cab}[\hat{\Gamma}] = \mathcal{R}^d{}_{cab}[\check{\Gamma}] + \check{\nabla}_a \mathcal{C}^d_{cb} - \check{\nabla}_b \mathcal{C}^d_{ca} + \mathcal{C}^e_{cb}\mathcal{C}^d_{ea} - \mathcal{C}^e_{ca}\mathcal{C}^d_{eb} \tag{4.80}$$

A *metric theory of gravity* has an action depending solely on a metric in a spacetime manifold, $S \equiv S[g]$. For completeness, instead of simply writing the Einstein-Hilbert action S_{EH} we will consider

$$S = \frac{1}{16\pi G}\int d^D x\, f(|g|)\mathcal{R}[\hat{\Gamma}]\ , \tag{4.81}$$

invariant under UCTs, that reduces to S_{EH} if $f(|g|) = \sqrt{-|g|}$ (only in this case we have invariance under GCTs). Now, we substitute $\mathcal{R} \equiv g^{cb}\mathcal{R}_{cb} \equiv g^{cb}\mathcal{R}^a{}_{cab}$ as given by (4.80):

$$S = \frac{1}{16\pi G}\int d^D x\Big[f(|g|)g^{cb}\mathcal{R}^a{}_{cab}[\check{\Gamma}] + f(|g|)g^{cb}\left(\check{\nabla}_a \mathcal{C}^a_{cb} - \check{\nabla}_b \mathcal{C}^a_{ca}\right) + \mathfrak{L}_1\Big] \tag{4.82}$$

where $\mathfrak{L}_1 = f(|g|)g^{cb}\left(\mathcal{C}^d_{cb}\mathcal{C}^a_{da} - \mathcal{C}^d_{ca}\mathcal{C}^a_{db}\right)$. Note that

$$\begin{aligned} &\int d^D x f(|g|)g^{cb}\left(\check{\nabla}_a \mathcal{C}^a_{cb} - \check{\nabla}_b \mathcal{C}^a_{ca}\right) \\ &\qquad = \int d^D x \check{\nabla}_d \mathfrak{B}^d - \int d^D x\Big[\mathcal{C}^a_{cb}\check{\nabla}_a\left(f(|g|)g^{cb}\right) - \mathcal{C}^a_{ca}\check{\nabla}_b\left(f(|g|)g^{cb}\right)\Big] \end{aligned} \tag{4.83}$$

where $\mathfrak{B}^d = f(|g|)\left(g^{cb}\mathcal{C}^d_{cb} - g^{dc}\mathcal{C}^a_{ca}\right)$. Note also that

$$\begin{aligned} \hat{\nabla}_a\left(\sqrt{-|g|}\right)^\omega &= \partial_a\left(\sqrt{-|g|}\right)^\omega - \omega\hat{\Gamma}^b_{ba}\left(\sqrt{-|g|}\right)^\omega \\ &= \omega\left(\sqrt{-|g|}\right)^{\omega-1}\partial_a\sqrt{-|g|} - \omega\left(\sqrt{-|g|}\right)^{\omega-1}\hat{\Gamma}^b_{ba}\sqrt{-|g|} = 0 \end{aligned} \tag{4.84}$$

since $\hat{\Gamma}^b_{ba}\sqrt{-|g|} = \partial_a\sqrt{-|g|}$. Hence, writing $f(|g|)$ as a polynomial, one sees that $\hat{\nabla}_a\left(f(|g|)g^{cb}\right) = 0$. Using this together with (4.76), one can write the last integral in (4.83) as

$$\begin{aligned} \mathcal{C}^a_{cb}\check{\nabla}_a\left[f(|g|)g^{cb}\right] - \mathcal{C}^a_{ca}\check{\nabla}_b\left[f(|g|)g^{cb}\right] = (\omega+1)g^{cb}\mathcal{C}^d_{cb}\mathcal{C}^a_{ad}f(|g|) - 2g^{cb}\mathcal{C}^d_{ca}\mathcal{C}^a_{db}f(|g|) \\ -(\omega-1)g^{cb}\mathcal{C}^d_{dc}\mathcal{C}^a_{ab}f(|g|) \end{aligned} \tag{4.85}$$

where we have assumed that $f(|g|) \equiv f^\omega(|g|) \propto \left(\sqrt{-|g|}\right)^\omega$, such that $\omega = 1$ corresponds to S_{EH} (in this case, (4.85) equals $2\mathfrak{L}_1$).

Chosing $\check{\Gamma}$ such that $\check{\Gamma}^b_{ba} = 0$ (which is unnecessary if $\omega = 1$) and we have $\check{\nabla}_d\mathfrak{B}^d = \partial_d\mathfrak{B}^d$,

$$S = \frac{1}{16\pi G}\int d^D x\Big(\partial_d\mathfrak{B}^d + f^\omega(|g|)g^{cb}\mathcal{R}^a{}_{cab}[\check{\Gamma}] - \mathfrak{L}_\omega\Big) \tag{4.86}$$

where

$$\begin{aligned}\mathfrak{L}_\omega &= \Big[\omega g^{cb}\mathcal{C}^d_{cb}\mathcal{C}^a_{ad} - g^{cb}\mathcal{C}^d_{ca}\mathcal{C}^a_{db} - (\omega-1)g^{cb}\mathcal{C}^d_{dc}\mathcal{C}^a_{ab}\Big] f^\omega(|g|) \\ &= \mathfrak{L}_1 + (\omega-1)g^{cb}\mathcal{C}^d_{cb}\mathcal{C}^a_{ad}f^\omega(|g|) - (\omega-1)g^{cb}\mathcal{C}^d_{dc}\mathcal{C}^a_{ab}f^\omega(|g|)\end{aligned} \tag{4.87}$$

Let us drop the total divergence in (4.86):

$$S \equiv \frac{1}{16\pi G}\int d^Dx\Big(f^\omega(|g|)g^{cb}\mathcal{R}^a{}_{cab}[\check{\Gamma}] - \mathfrak{L}_\omega\Big)\,. \tag{4.88}$$

Most commonly, the Riemann in S_{EH} is written in terms of the metric connection $\hat{\Gamma}[g]$ (and coordinate derivatives ∂). Have we done this instead of (4.80), we would end up with

$$S_{\text{EH}} = \frac{1}{16\pi G}\int d^Dx\left(\partial_d\mathfrak{B}'^d - \mathfrak{L}'_1\right) \tag{4.89}$$

in place of (4.86) (with $\omega = 1$). $\mathfrak{L}'$ and $\mathfrak{B}'$ only differ from $\mathfrak{L}$ and $\mathfrak{B}$ in having $\hat{\Gamma}$ instead of $\mathcal{C}$ but they aren't scalar and vector densities. Only S_{EH} is invariant under GCTs. Fortunately in this "bi-connection" formulation this is not the case and we can drop the total divergence in (4.86) while mantaining the action a scalar.

Using (4.78), one sees that

$$g^{db}\mathcal{C}^c_{da}\mathcal{C}^a_{bc} = \frac{1}{4}\left[2\delta^c_{(e}g_{f)(a}\delta^d_{b)} - g^{cd}g_{a(e}g_{f)b}\right]\check{\nabla}_c g^{ab}\check{\nabla}_d g^{ef} \tag{4.90}$$

$$g^{cb}\mathcal{C}^d_{cb}\mathcal{C}^a_{ad} = \frac{1}{4}\left[\delta^c_{(e}\delta^d_{f)}g_{ab} + \delta^c_{(a}\delta^d_{b)}g_{ef} - g^{cd}g_{ab}g_{ef}\right]\check{\nabla}_c g^{ab}\check{\nabla}_d g^{ef} \tag{4.91}$$

$$g^{cb}\mathcal{C}^d_{dc}\mathcal{C}^a_{ab} = \frac{1}{4}g^{cd}g_{ab}g_{ef}\check{\nabla}_c g^{ab}\check{\nabla}_d g^{ef} \tag{4.92}$$

and then

$$\begin{aligned}\mathfrak{L}_\omega &= \frac{f^\omega(|g|)}{4}\left[g^{cd}g_{a(e}g_{f)b} - (2\omega-1)g^{cd}g_{ab}g_{ef} - 2\delta^c_{(e}g_{f)(a}\delta^d_{b)} + \omega\delta^c_{(e}\delta^d_{f)}g_{ab} + \omega\delta^c_{(a}\delta^d_{b)}g_{ef}\right]\check{\nabla}_c g^{ab}\check{\nabla}_d g^{ef} \\ &= \mathfrak{L}_1 + \frac{\omega-1}{4}f^\omega(|g|)\left[\delta^c_{(e}\delta^d_{f)}g_{ab} + \delta^c_{(a}\delta^d_{b)}g_{ef} - 2g^{cd}g_{ab}g_{ef}\right]\check{\nabla}_c g^{ab}\check{\nabla}_d g^{ef} \\ &\equiv \frac{f^\omega(|g|)}{2}K^\omega{}_{ab}{}^c{}_{ef}{}^d[g]\check{\nabla}_c g^{ab}\check{\nabla}_d g^{ef}\end{aligned} \tag{4.93}$$

where

$$\mathfrak{L}_1 = \frac{f^\omega(|g|)}{4}\left[g^{cd}g_{a(e}g_{f)b} - g^{cd}g_{ab}g_{ef} - 2\delta^c_{(e}g_{f)(a}\delta^d_{b)} + \delta^c_{(e}\delta^d_{f)}g_{ab} + \delta^c_{(a}\delta^d_{b)}g_{ef}\right]\check{\nabla}_c g^{ab}\check{\nabla}_d g^{ef} \tag{4.94}$$

From now on until the end of the section, we'll focus solely on the Einstein-Hilbert case, where $\omega = 1$ and there's no need for $\check{\Gamma}^b_{ba} = 0$. One sees that $2K^1{}_{ab}{}^c{}_{ef}{}^d[g]$ (the expression inside the brackets above) equals $2K_{ab}{}^{c}{}_{ef}{}^{d}[g]$ from (4.24). By choosing the connection $\check{\Gamma}$ to be flat ($\check{\Gamma} \equiv \Gamma \Rightarrow \check{\nabla} \equiv \bar{\nabla}$), such that $\mathcal{R}^{a}{}_{cab}[\Gamma] = 0$, one obtains from (4.88)

$$S_{\text{EH}} = \frac{-1}{32\pi G} \int d^D x \sqrt{-|g|} K_{ab}{}^{c}{}_{ef}{}^{d}[g] \bar{\nabla}_c g^{ab} \bar{\nabla}_d g^{ef} \tag{4.95}$$

and claim A is therefore proven. The action S_h given by (4.25) is the first non-vanishing term in the taylor expansion $S_{\text{EH}}|_{g^{ab}=\bar{g}^{ab}+\sqrt{16\pi G}h^{ab}}$, i.e. around a flat metric (such that $\check{\Gamma}$ can be the Levi-Civita connection associated with $\bar{g}$).

Let us for a moment consider the term with the Riemman in S_{EH} given by (4.88) and explore it by not assuming a flat connection $\check{\Gamma}$:

$$\frac{1}{16\pi G} \int d^D x \sqrt{-|g|} g^{cb} \mathcal{R}^{a}{}_{cab}[\check{\Gamma}] \equiv S_{\mathcal{R}}[g] \neq 0 \tag{4.96}$$

Choosing $\check{\Gamma}$ to be the Levi-Civita connection associated with an arbitrary metric γ ($\check{\Gamma} = \check{\Gamma}[\gamma]$), a (functional) Taylor expansion about γ gives

$$\begin{aligned} &S_{\mathcal{R}}|_{g^{ab}=\gamma^{ab}+\sqrt{16\pi G}h^{ab}} \\ &\quad= \sum_{n=0}^{\infty} \frac{(16\pi G)^{\frac{n-2}{2}}}{n!} \int d^D x\, d^D x_1 \ldots d^D x_n \frac{\delta^n(\sqrt{-|\gamma|}\gamma^{cb})}{\delta\gamma^{ab}(x_1)\ldots\delta\gamma^{ij}(x_n)} h^{ab}(x_1)\ldots h^{ij}(x_n) \mathcal{R}^{a}{}_{cab}[\gamma] \\ &\quad\equiv \sum_{n=0}^{\infty} (16\pi G)^{\frac{n-2}{2}} S_{\mathcal{R}}^{(n)} \end{aligned} \tag{4.97}$$

Computing $S_{\mathcal{R}}^{(2)}$, one sees that it equals S_0^{NM} (4.63). Hence, using (4.70) and (4.74), we get that $S_{\mathcal{R}}^{(n)} = S_{n-2}^{\text{NM}}$ (up to an overall constant) for $n \geq 2$. For completeness, let us do the same for the non Einstein-Hilbert case ($\omega \neq 1$). Then, instead of (4.96) we have

$$\frac{1}{16\pi G} \int d^D x\, f^{\omega}(|g|) g^{cb} \mathcal{R}^{a}{}_{cab}[\check{\Gamma}] \equiv S_{U,\mathcal{R}}[g] \tag{4.98}$$

and, expanding this around γ,

$$\begin{aligned} &S_{U,\mathcal{R}}|_{g^{ab}=\gamma^{ab}+\sqrt{16\pi G}h^{ab}} \\ &\quad= \sum_{n=0}^{\infty} \frac{(16\pi G)^{\frac{n-2}{2}}}{n!} \int d^D x\, d^D x_1 \ldots d^D x_n \frac{\delta^n(f^{\omega}(|\gamma|)\gamma^{cb})}{\delta\gamma^{ab}(x_1)\ldots\delta\gamma^{ij}(x_n)} h^{ab}(x_1)\ldots h^{ij}(x_n) \mathcal{R}^{a}{}_{cab}[\gamma] \\ &\quad\equiv \sum_{n=0}^{\infty} (16\pi G)^{\frac{n-2}{2}} S_{U,\mathcal{R}}^{(n)} \end{aligned} \tag{4.99}$$

which will be useful in section 5.2.

4.4 Reverse engineering Einstein's gravity

We define *background independence* of a theory as the absence of *absolute objects* (see [21]) in its formulation. However, it's quite common to see this, in the literature, being included in the concept of *general covariance* which we find unnecessarily misleading, specially in the context of this work. In this section, we're going to see how background independence of S_{EH}[21] leads to:

- the gauge invariance of S_h and consequently $\mathscr{A}_h$ in the Diff case;
- and the self-coupling condition (4.21).

Given a metric theory $S[g]$, one can "divide" the (components of the inverse) metric g^{ab} into a non-dynamical background metric $\breve{g}^{ab}$ and a dynamical h^{ab} field:

$$S[g]\Big|_{g^{ab}=\phi(\chi h^{ab},\breve{g}^{ab})} \equiv S[\breve{g};h] = \sum_{n=0}^{\infty} \chi^n S^{(n)}[\breve{g};h] \tag{4.100}$$

We assume the background is flat, $\breve{g}^{ab} = \bar{g}^{ab}$, and

$$S[g] \equiv (16\pi G) S_{\text{EH}} = \frac{-1}{2} \int d^D x \, \sqrt{-|g|} K_{ab}{}^{c}{}_{ef}{}^{d}[g] \bar{\nabla}_c g^{ab} \bar{\nabla}_d g^{ef} \tag{4.101}$$

such that, considering $\phi = \bar{g}^{ab} + \chi h^{ab}$:

$$\begin{aligned} S[\bar{g};h] &= \sum_{n=2}^{\infty} \frac{\chi^n}{n!} \int d^D x_1 ... \, d^D x_n \frac{\delta^n S[\gamma]}{\delta\gamma^{ab}(x_1) ... \, \delta\gamma^{ij}(x_n)}\Big|_{\gamma=\bar{g}} h^{ab}(x_1) ... \, h^{ij}(x_n) \\ &\equiv \sum_{n=0}^{\infty} \chi^n S_n[h;\bar{g}] \ . \end{aligned} \tag{4.102}$$

(Metric compatibility of $\bar{\nabla}$ leads to $S^{(0)} = 0 = S^{(1)}$.) Note that, since $S[g = \phi] = S[\bar{g};h]$ is a scalar, S_n are too. However for the moment we focus on the background independent action $S[g]$. Since this is a scalar, under GCTs in the active perspective we have (up to ST)

$$\delta_\xi S[g] \equiv S[g^{ab} + \delta_\xi g^{ab}] - S[g^{ab}] = 0 \quad \text{with} \quad \delta_\xi g^{ab} = \xi^c \bar{\nabla}_c g^{ab} - 2 g^{c(a} \bar{\nabla}_c \xi^{b)} \ .$$

where (3.21) was used. Note that

$$\delta_\xi g^{ab}\Big|_{g^{ab}=\phi} = -2\bar{\nabla}^{(a}\xi^{b)} + \chi \xi^c \bar{\nabla}_c h^{ab} - 2\chi h^{c(a} \bar{\nabla}_c \xi^{b)}$$

[21] The footnotes in p. 5 of [38] and references therein suggest that if this label is used thoroughly it doesn't apply to GR.

such that, using metric compatibility,

$$\delta_\xi S[h;\bar{g}] = 0 \quad \text{with} \quad \delta_\xi \bar{g}^{ab} + \chi \delta_\xi h^{ab} = -2\bar{\nabla}^{(a}\xi^{b)} + \chi \xi^c \bar{\nabla}_c h^{ab} - 2\chi h^{c(a}\bar{\nabla}_c \xi^{b)} \ .$$

Since the h^{ab} field is the dynamical one, we choose $\delta_\xi \bar{g}^{ab} = 0$ and $\delta_\xi h^{ab} = \chi^{-1}\delta_\xi^{(-1)} h^{ab} + \delta_\xi^{(0)} h^{ab}$ where

$$\begin{aligned} \delta_\xi^{(-1)} h^{ab} &= -2\bar{\nabla}^{(a}\xi^{b)} \\ \delta_\xi^{(0)} h^{ab} &= \xi^c \bar{\nabla}_c h^{ab} - 2h^{c(a}\bar{\nabla}_c \xi^{b)} \end{aligned} \tag{4.103}$$

Note that all functional integrals should be automatically understood to have $\bar{g}$ and h as variables.

$$\begin{aligned} \delta_\xi S_n = \int d^D x \, \frac{\delta S_n}{\delta h^{ab}} \delta_\xi h^{ab} + ST &= \int d^D x \left(\chi^{-1} \frac{\delta S_n}{\delta h^{ab}} \delta_\xi^{(-1)} h^{ab} + \frac{\delta S_n}{\delta h^{ab}} \delta_\xi^{(0)} h^{ab} \right) + ST \\ &\equiv \chi^{-1} \delta_\xi^{(-1)} S_n + \delta_\xi^{(0)} S_n \end{aligned} \tag{4.104}$$

Hence, since S is invariant (up to ST) under GCTs:

$$\delta_\xi^{(-1)} S_0 + ST = 0 \tag{4.105}$$

$$\delta_\xi^{(0)} S_{n-1} + \delta_\xi^{(-1)} S_n + ST = 0 \ , \quad n \geq 1 \tag{4.106}$$

Using (4.105),

$$\delta_\xi^{(-1)} S_0 = -2 \int d^D x \, \frac{\delta S_0}{\delta h^{ab}} \bar{\nabla}^a \xi^b + ST \Rightarrow 2 \int d^D x \, \xi^b \bar{\nabla}^a \frac{\delta S_0}{\delta h^{ab}} + ST = 0 \tag{4.107}$$

We can convert the surface terms into an integral over the boundary and, using an appropriate choice of boundary conditions for the gauge parameter ξ, it can be made to vanish. Due to arbitrariness of the parameter one arrives at the identity

$$\bar{\nabla}^a \frac{\delta S_0}{\delta h^{ab}} = 0 \tag{4.108}$$

Moving on to $n \geq 1$, one has that

$$\begin{aligned} \delta_\xi^{(-1)} S_n &= \int d^D x \, \frac{\delta S_n}{\delta h^{ab}} \delta_\xi^{(-1)} h^{ab} + ST = -2 \int d^D x \, \frac{\delta S_n}{\delta h^{ab}} \bar{\nabla}^a \xi^b + ST \\ \delta_\xi^{(0)} S_{n-1} &= \int d^D x \, \frac{\delta S_{n-1}}{\delta h^{ab}} \delta_\xi^{(0)} h^{ab} + ST \\ &= \int d^D x \left(-2 \frac{\delta S_{n-1}}{\delta h^{ab}} h^{ca} \bar{\nabla}_c \xi^b + \frac{\delta S_{n-1}}{\delta h^{ab}} \xi^c \bar{\nabla}_c h^{ab} \right) + ST \ . \end{aligned} \tag{4.109}$$

Using (4.106) and integrating by parts to get rid of $\bar{\nabla}\xi$, one obtains

$$\int d^D x \left[2\bar{\nabla}^c \frac{\delta S_n}{\delta h^{cd}} + 2\bar{\nabla}_c \left(\frac{\delta S_{n-1}}{\delta h^{ad}} h^{ca} \right) + \frac{\delta S_{n-1}}{\delta h^{ab}} \bar{\nabla}_d h^{ab} \right] \xi^d + ST = 0 \tag{4.110}$$

It's time to use the invariance of S_n under GCTs. This is where, as promised, (3.37) (without φ fields) enters and we rewrite the equation above as

$$\int d^D x \left[2\bar{\nabla}^c \frac{\delta S_n}{\delta h^{cd}} - 2\bar{\nabla}^c \frac{\delta S_{n-1}}{\delta \bar{g}^{cd}}\right] \xi^d + ST = 0 \tag{4.111}$$

Following the same reasoning used to derive (4.108), we obtain the identity

$$\bar{\nabla}^a \frac{\delta S_n}{\delta h^{ab}} = \bar{\nabla}^a \frac{\delta S_{n-1}}{\delta \bar{g}^{ab}} \tag{4.112}$$

If one recalls section 4.1, namely equation (4.16), one sees that this is equivalent to the self-coupling condition (4.21). One could ask: what if $\breve{g}$ wasn't flat and $S[g]$ wasn't of the type (4.101) such that $S^{(0)}$ and $S^{(1)}$ didn't vanish? Then, instead of (4.105) and (4.106), we would have

$$\delta_\xi^{(-1)} S^{(0)} + ST = 0 \tag{4.113}$$

$$\delta_\xi^{(0)} S^{(n-1)} + \delta_\xi^{(-1)} S^{(n)} + ST = 0\ , \quad n \geq 1 \tag{4.114}$$

where, instead of (4.103),

$$\begin{aligned} \delta_\xi^{(-1)} h^{ab} &= -2\breve{\nabla}^{(a}\xi^{b)} \\ \delta_\xi^{(0)} h^{ab} &= \xi^c \breve{\nabla}_c h^{ab} - 2h^{c(a}\breve{\nabla}_c \xi^{b)}\ . \end{aligned} \tag{4.115}$$

($\breve{\nabla}$ is compatible with $\breve{g}$). Since $S^{(0)} = S^{(0)}[\breve{g}]$, $\delta_\xi^{(-1)} S^{(0)} = 0$ and $\delta_\xi^{(0)} S^{(0)} = 0$ are trivial. Note that

$$S^{(1)} = \chi \int d^D x \frac{\delta S[\gamma]}{\delta \gamma^{ab}}\bigg|_{\gamma=\breve{g}} h^{ab} \Rightarrow \frac{\delta S^{(1)}}{\delta h^{ab}} = \chi \frac{\delta S[\gamma]}{\delta \gamma^{ab}}\bigg|_{\gamma=\breve{g}} \tag{4.116}$$

Then, if the background is a solution of the EOMs associated with $S[g]$, $\delta_\xi^{(-1)} S^{(1)} = 0 = \delta_\xi^{(0)} S^{(1)}$ and (4.114) reduce to

$$\delta_\xi^{(-1)} S^{(2)} + ST = 0 \tag{4.117}$$

$$\delta_\xi^{(0)} S^{(n-1)} + \delta_\xi^{(-1)} S^{(n)} + ST = 0\ , \quad n \geq 3 \tag{4.118}$$

similar to (4.105) and (4.106). On the other hand, if the background is not a solution, the action (of quadratic order in the dynamical field) $S^{(2)}$ doesn't have any invariance besides the one under GCTs.

To end this section, let us consider a metric theory of gravity with matter, given by the action S_{mat}.

$$\begin{aligned} S_{\text{mat}}[g,\varphi]\Big|_{g=\phi} &= \sum_{n=0}^{\infty} \frac{\chi^n}{n!} \int d^D x_1 ... \, d^D x_n \frac{\delta^n S_{\text{mat}}[\gamma,\varphi]}{\delta\gamma^{ab}(x_1)... \, \delta\gamma^{ij}(x_n)}\Bigg|_{\gamma=\bar{g}} h^{ab}(x_1) ... \, h^{ij}(x_n) \\ &\equiv S_{\text{mat}}[\bar{g};h,\varphi] = \sum_{n=0}^{\infty} \chi^n S^{(n)}_{\text{mat}}[\bar{g};h,\varphi] \end{aligned} \tag{4.119}$$

Under GCTs in the active perspective we have (up to ST)

$$\delta_\xi S_{\text{mat}}[g,\varphi] = 0 \quad \text{with} \quad \delta_\xi g^{ab} = \xi^c \bar{\nabla}_c g^{ab} - 2g^{c(a}\bar{\nabla}_c \xi^{b)}$$
$$\text{and} \quad \delta_\xi \varphi = \xi^c \bar{\nabla}_c \varphi$$

Note that

$$\delta_\xi g^{ab}\Big|_{g^{ab}=\phi} = -2\bar{\nabla}^{(a}\xi^{b)} + \chi \xi^c \bar{\nabla}_c h^{ab} - 2\chi h^{c(a}\bar{\nabla}_c \xi^{b)}$$

such that

$$\delta_\xi S_{\text{mat}}[\bar{g};h,\varphi] = 0 \quad \text{with} \quad \delta_\xi \bar{g}^{ab} + \chi \delta_\xi h^{ab} = -2\bar{\nabla}^{(a}\xi^{b)} + \chi \xi^c \bar{\nabla}_c h^{ab} - 2\chi h^{c(a}\bar{\nabla}_c \xi^{b)}$$
$$\text{and} \quad \delta_\xi \varphi = \xi^c \bar{\nabla}_c \varphi \equiv \delta^{(0)}_\xi \varphi$$

Like we did above, we choose $\delta_\xi \bar{g}^{ab} = 0$ and $\delta_\xi h^{ab} = \chi^{-1}\delta^{(-1)}_\xi h^{ab} + \delta^{(0)}_\xi h^{ab}$. Note that all functional integrals should be automatically understood to have $\bar{g}$, h and φ as variables.

$$\begin{aligned} \delta_\xi S^{(n)}_{\text{mat}} &= \int d^D x \left(\frac{\delta S^{(n)}_{\text{mat}}}{\delta h^{ab}} \delta_\xi h^{ab} + \frac{\delta S^{(n)}_{\text{mat}}}{\delta \varphi} \delta_\xi \varphi \right) + ST \\ &= \int d^D x \left(\chi^{-1} \frac{\delta S^{(n)}_{\text{mat}}}{\delta h^{ab}} \delta^{(-1)}_\xi h^{ab} + \frac{\delta S^{(n)}_{\text{mat}}}{\delta h^{ab}} \delta^{(0)}_\xi h^{ab} + \frac{\delta S^{(n)}_{\text{mat}}}{\delta \varphi} \delta^{(0)}_\xi \varphi \right) + ST \\ &\equiv \chi^{-1}\delta^{(-1)}_\xi S^{(n)}_{\text{mat}} + \delta^{(0)}_\xi S^{(n)}_{\text{mat}} \end{aligned} \tag{4.120}$$

Hence, since S_{mat} is invariant (up to ST) under GCTs:

$$\delta^{(-1)}_\xi S^{(0)}_{\text{mat}} + ST = 0 \tag{4.121}$$

$$\delta^{(0)}_\xi S^{(n-1)}_{\text{mat}} + \delta^{(-1)}_\xi S^{(n)}_{\text{mat}} + ST = 0 \, , \quad n \geq 1 \tag{4.122}$$

The first equation is trivially satisfied since $S^{(0)}_{\text{mat}}$ has no dependence on h. Furthermore, one has

$$
\begin{aligned}
\delta_\xi^{(-1)} S_{\text{mat}}^{(n)} &= \int d^D x\, \frac{\delta S_{\text{mat}}^{(n)}}{\delta h^{ab}} \delta_\xi^{(-1)} h^{ab} + ST = -2 \int d^D x\, \frac{\delta S_{\text{mat}}^{(n)}}{\delta h^{ab}} \bar{\nabla}^a \xi^b + ST \\
\delta_\xi^{(0)} S_{\text{mat}}^{(n-1)} &= \int d^D x \left(\frac{\delta S_{\text{mat}}^{(n-1)}}{\delta h^{ab}} \delta_\xi^{(0)} h^{ab} + \frac{\delta S_{\text{mat}}^{(n-1)}}{\delta \varphi} \delta_\xi^{(0)} \varphi \right) + ST \\
&= \int d^D x \left(-2 \frac{\delta S_{\text{mat}}^{(n-1)}}{\delta h^{ab}} h^{ca} \bar{\nabla}_c \xi^b + \frac{\delta S_{\text{mat}}^{(n-1)}}{\delta h^{ab}} \xi^c \bar{\nabla}_c h^{ab} + \frac{\delta S_{\text{mat}}^{(n-1)}}{\delta \varphi} \xi^c \bar{\nabla}_c \varphi \right) + ST \, .
\end{aligned}
\tag{4.123}
$$

Using (4.122) and integrating by parts, one obtains

$$
\int d^D x \left[2 \bar{\nabla}^c \frac{\delta S_{\text{mat}}^{(n)}}{\delta h^{cd}} + 2 \bar{\nabla}_c \left(\frac{\delta S_{\text{mat}}^{(n-1)}}{\delta h^{ad}} h^{ca} \right) + \frac{\delta S_{\text{mat}}^{(n-1)}}{\delta h^{ab}} \bar{\nabla}_d h^{ab} + \frac{\delta S_{\text{mat}}^{(n-1)}}{\delta \varphi} \bar{\nabla}_d \varphi \right] \xi^d + ST = 0
\tag{4.124}
$$

Using (3.37) again (this time with φ fields), we write this as

$$
\int d^D x \left[2 \bar{\nabla}^c \frac{\delta S_{\text{mat}}^{(n)}}{\delta h^{cd}} - 2 \bar{\nabla}^c \frac{\delta S_{\text{mat}}^{(n-1)}}{\delta \bar{g}^{cd}} \right] \xi^d + ST = 0
\tag{4.125}
$$

Following the same reasoning used to derive (4.108), we obtain the identity

$$
\bar{\nabla}^a \frac{\delta S_{\text{mat}}^{(n)}}{\delta h^{ab}} = \bar{\nabla}^a \frac{\delta S_{\text{mat}}^{(n-1)}}{\delta \bar{g}^{ab}}
\tag{4.126}
$$

This is equivalent to (4.19). It also implies

$$
\begin{aligned}
\bar{\nabla}^a \sum_{n=1}^{\infty} \chi^n \frac{\delta S_{\text{mat}}^{(n)}}{\delta h^{ab}} = \bar{\nabla}^a \sum_{n=1}^{\infty} \chi^n \frac{\delta S_{\text{mat}}^{(n-1)}}{\delta \bar{g}^{ab}} &\Leftrightarrow \bar{\nabla}^a \frac{\delta S_{\text{mat}}}{\delta h^{ab}} - \bar{\nabla}^a \frac{\delta S_{\text{mat}}^{(0)}}{\delta h^{ab}} = \chi \bar{\nabla}^a \frac{\delta S_{\text{mat}}}{\delta \bar{g}^{ab}} \\
&\Leftrightarrow \bar{\nabla}^a \frac{\delta S_{\text{mat}}}{\delta h^{ab}} = \bar{\nabla}^a \frac{\delta S_{\text{mat}}^{(0)}}{\delta h^{ab}} + \frac{\chi}{2} \bar{\nabla}_c T_{\text{Ros}}^{cd} \bar{g}_{db} \sqrt{-|\bar{g}|}
\end{aligned}
\tag{4.127}
$$

which coincides with (2.45) in a covariantized form.

5 Self-coupling in the TDiff case

5.1 Reverse engineering Unimodular gravity

In section 4.3, besides GR, we considered other metric theories (given by $S[g]$) that, whilst not covariant, enjoyed U-covariance (cf. [22, 23]). Close to what was done in the previous section, we "divide" g^{ab} into a non-dynamical (flat and "unimodular") background metric $\bar{\mathfrak{g}}^{ab}$ and a dynamical h^{ab} field:

$$S[g]\Big|_{g^{ab}=\phi(\chi h^{ab},\bar{\mathfrak{g}}^{ab})} \equiv S_U[\bar{\mathfrak{g}};h] = \sum_{n=0}^{\infty} \chi^n S_U^{(n)}[\bar{\mathfrak{g}};h] \tag{5.1}$$

Taking (4.88) and (4.93) into consideration, we choose the connection $\check{\Gamma}$ such that $\mathcal{R}^a{}_{bcd}[\check{\Gamma}] = 0$ and $\check{\Gamma}^a_{ab} = 0$ ($\check{\Gamma} \equiv \mathbf{\Gamma} \Rightarrow \check{\nabla} \equiv \bar{\boldsymbol{\nabla}}$). Hence,

$$S[g] \equiv \frac{-1}{2}\int d^D x\, f(|g|) K_{ab}{}^c{}_{ef}{}^d[g] \bar{\boldsymbol{\nabla}}_c g^{ab} \bar{\boldsymbol{\nabla}}_d g^{ef} \tag{5.2}$$

Assuming $\phi = \bar{\mathfrak{g}}^{ab} + \chi h^{ab}$,

$$\begin{aligned} S_U[\bar{\mathfrak{g}};h] &= \sum_{n=2}^{\infty} \frac{\chi^n}{n!} \int d^D x_1 \ldots d^D x_n \frac{\delta^n S[\gamma]}{\delta\gamma^{ab}(x_1)\ldots\delta\gamma^{ij}(x_n)}\Bigg|_{\gamma=\bar{\mathfrak{g}}} h^{ab}(x_1)\ldots h^{ij}(x_n) \\ &\equiv \sum_{n=0}^{\infty} \chi^n S_{U,n}[\bar{\mathfrak{g}};h]\ . \end{aligned} \tag{5.3}$$

(Metric compatibility of $\bar{\boldsymbol{\nabla}}$ leads to $S_U^{(0)} = 0 = S_U^{(1)}$.) This time we only assume $S[g=\phi] = S[\bar{\mathfrak{g}};h]$ to be a U-scalar, such that S_n are too U-scalars. However let us focus for now on the background independent $S[g]$. Under UCTs in the active perspective we have (up to ST)

$$\delta_{\boldsymbol{\xi}} S[g] = 0 \quad \text{with} \quad \delta_{\boldsymbol{\xi}} g^{ab} = \boldsymbol{\xi}^c \partial_c g^{ab} - 2g^{c(a}\partial_c \boldsymbol{\xi}^{b)} = \boldsymbol{\xi}^c \bar{\boldsymbol{\nabla}}_c g^{ab} - 2g^{c(a}\bar{\boldsymbol{\nabla}}_c \boldsymbol{\xi}^{b)}\ .$$

Note that

$$\delta_{\boldsymbol{\xi}} g^{ab}\Big|_{g^{ab}=\phi} = -2\bar{\boldsymbol{\nabla}}^{(a}\boldsymbol{\xi}^{b)} + \chi\boldsymbol{\xi}^c \bar{\boldsymbol{\nabla}}_c h^{ab} - 2\chi h^{c(a}\bar{\boldsymbol{\nabla}}_c \boldsymbol{\xi}^{b)}$$

such that (again up to ST)

$$\delta_{\boldsymbol{\xi}} S_U[\bar{\mathfrak{g}};h] = 0 \quad \text{with} \quad \delta_{\boldsymbol{\xi}}\bar{\mathfrak{g}}^{ab} + \chi\delta_{\boldsymbol{\xi}} h^{ab} = -2\bar{\boldsymbol{\nabla}}^{(a}\boldsymbol{\xi}^{b)} + \chi\boldsymbol{\xi}^c \bar{\boldsymbol{\nabla}}_c h^{ab} - 2\chi h^{c(a}\bar{\boldsymbol{\nabla}}_c \boldsymbol{\xi}^{b)}\ .$$

Since the h^{ab} field is the dynamical one, we choose $\delta_{\boldsymbol{\xi}}\bar{\mathfrak{g}}^{ab} = 0$ and $\delta_{\boldsymbol{\xi}} h^{ab} \equiv \chi^{-1}\delta_{\boldsymbol{\xi}}^{(-1)} h^{ab} + \delta_{\boldsymbol{\xi}}^{(0)} h^{ab}$ where

$$\begin{aligned} &\delta_{\boldsymbol{\xi}}^{(-1)} h^{ab} = -2\bar{\boldsymbol{\nabla}}^{(a}\boldsymbol{\xi}^{b)} \Rightarrow \bar{\mathfrak{g}}_{ab}\delta_{\boldsymbol{\xi}}^{(-1)} h^{ab} = -2\bar{\boldsymbol{\nabla}}_a \boldsymbol{\xi}^a = 0 \\ &\delta_{\boldsymbol{\xi}}^{(0)} h^{ab} = \boldsymbol{\xi}^c \bar{\boldsymbol{\nabla}}_c h^{ab} - 2h^{c(a}\bar{\boldsymbol{\nabla}}_c \boldsymbol{\xi}^{b)} \end{aligned} \tag{5.4}$$

Note that all functional integrals should be automatically understood to have $\bar{\mathfrak{g}}$ and h as variables. Like in previous section we write $\delta_{\boldsymbol{\xi}} S_{U,n} = \chi^{-1}\delta^{(-1)}_{\boldsymbol{\xi}} S_{U,n} + \delta^{(0)}_{\boldsymbol{\xi}} S_{U,n}$ where

$$\begin{aligned} \delta^{(-1)}_{\boldsymbol{\xi}} S_{U,n} &\equiv \int d^D x \, \frac{\delta S_{U,n}}{\delta h^{ab}} \delta^{(-1)}_{\boldsymbol{\xi}} h^{ab} \\ \delta^{(0)}_{\boldsymbol{\xi}} S_{U,n} &\equiv \int d^D x \, \frac{\delta S_{U,n}}{\delta h^{ab}} \delta^{(0)}_{\boldsymbol{\xi}} h^{ab} \end{aligned} \tag{5.5}$$

Hence, since S is invariant (up to ST) under UCTs:

$$\delta^{(-1)}_{\boldsymbol{\xi}} S_0 + ST = 0 \tag{5.6}$$

$$\delta^{(0)}_{\boldsymbol{\xi}} S_{U,n-1} + \delta^{(0)}_{\boldsymbol{\xi}} S_{U,n} + ST = 0 \, , \quad n \geq 1 \tag{5.7}$$

One has that

$$\delta^{(-1)}_{\boldsymbol{\xi}} S_0 = \int d^D x \, \frac{\delta S_0}{\delta h^{ab}} \delta^{(-1)}_{\boldsymbol{\xi}} h^{ab} + ST = -2 \int d^D x \, \frac{\delta S_0}{\delta h^{ab}} \bar{\boldsymbol{\nabla}}^a \boldsymbol{\xi}^b + ST \tag{5.8}$$

Integrating the first term by parts and using (5.6), one obtains

$$2 \int d^D x \, \boldsymbol{\xi}^b \bar{\boldsymbol{\nabla}}^a \frac{\delta S_0}{\delta h^{ab}} + ST = 0 \tag{5.9}$$

The transformation parameter is constrained by $\partial_a \boldsymbol{\xi}^a = \bar{\boldsymbol{\nabla}}_a \boldsymbol{\xi}^a = 0$ such that we can replace it by the more arbitrary parameter $\mathcal{F}^{cj} = \mathcal{F}^{[cj]}$: $\boldsymbol{\xi}^j = \partial_c \mathcal{F}^{cj} = \bar{\boldsymbol{\nabla}}_c \mathcal{F}^{cj}$. After doing this and integrating by parts,

$$-2 \int d^D x \, \mathcal{F}^{cb} \bar{\boldsymbol{\nabla}}_c \bar{\boldsymbol{\nabla}}^a \frac{\delta S_0}{\delta h^{ab}} + ST = 0 \tag{5.10}$$

We can convert the surface term above into an integral over the boundary and, using an appropriate choice of boundary conditions for the gauge parameter $\boldsymbol{\xi}^a$, it can be made to vanish. Due to arbitrariness of the parameter $\mathcal{F}^{cb}$ one arrives at the identity

$$0 = \bar{\boldsymbol{\nabla}}_{[c} \bar{\boldsymbol{\nabla}}^a \frac{\delta S_0}{\delta h^{b]a}} = \partial_{[c} \bar{\boldsymbol{\nabla}}^a \frac{\delta S_0}{\delta h^{b]a}} \tag{5.11}$$

Moving on to $n \geq 1$, one has that

$$\delta^{(-1)}_{\boldsymbol{\xi}} S_{U,n} = \int d^D x \, \frac{\delta S_{U,n}}{\delta h^{ab}} \delta^{(-1)}_{\boldsymbol{\xi}} h^{ab} + ST = -2 \int d^D x \, \frac{\delta S_{U,n}}{\delta h^{ab}} \bar{\boldsymbol{\nabla}}^a \boldsymbol{\xi}^b + ST \tag{5.12}$$

and

$$\begin{aligned} \delta^{(0)}_{\boldsymbol{\xi}} S_{U,n-1} &= \int d^D x \, \frac{\delta S_{U,n-1}}{\delta h^{ab}} \delta^{(0)}_{\boldsymbol{\xi}} h^{ab} + ST \\ &= \int d^D x \left(-2 \frac{\delta S_{U,n-1}}{\delta h^{ab}} h^{ca} \bar{\boldsymbol{\nabla}}_c \boldsymbol{\xi}^b + \frac{\delta S_{U,n-1}}{\delta h^{ab}} \boldsymbol{\xi}^c \bar{\boldsymbol{\nabla}}_c h^{ab} \right) + ST \, . \end{aligned} \tag{5.13}$$

Using (4.106) and integrating by parts to get rid of $\bar{\boldsymbol{\nabla}}\boldsymbol{\xi}$, one obtains

$$\int d^Dx \left[2\bar{\boldsymbol{\nabla}}^c \frac{\delta S_{U,n}}{\delta h^{cd}} + 2\bar{\boldsymbol{\nabla}}_c \left(\frac{\delta S_{U,n-1}}{\delta h^{ad}} h^{ca}\right) + \frac{\delta S_{U,n-1}}{\delta h^{ab}} \bar{\boldsymbol{\nabla}}_d h^{ab}\right] \boldsymbol{\xi}^d + ST = 0 \tag{5.14}$$

Inserting the more arbitrary parameter $\mathcal{F}^{id} = \mathcal{F}^{[id]}$ via $\boldsymbol{\xi}^d = \partial_i \mathcal{F}^{id} = \bar{\boldsymbol{\nabla}}_i \mathcal{F}^{id}$ and integrating by parts:

$$-\int d^Dx \left[2\bar{\boldsymbol{\nabla}}^c \bar{\boldsymbol{\nabla}}_{[i} \frac{\delta S_{U,n}}{\delta h^{d]c}} + 2\bar{\boldsymbol{\nabla}}_c \bar{\boldsymbol{\nabla}}_{[i} \left(\frac{\delta S_{U,n-1}}{\delta h^{d]a}} h^{ac}\right) + \bar{\boldsymbol{\nabla}}_{[i} \left(\left(\bar{\boldsymbol{\nabla}}_{d]} h^{ab}\right) \frac{\delta S_{U,n-1}}{\delta h^{ab}}\right)\right] \mathcal{F}^{id} + ST = 0 \tag{5.15}$$

It's time to use the invariance of $S_{U,n}$ under UCTs. This is where (3.62) (without φ fields) enters and we rewrite the equation above as

$$-2\int d^Dx \left[\bar{\boldsymbol{\nabla}}_{[i} \bar{\boldsymbol{\nabla}}^c \frac{\delta S_{U,n}}{\delta h^{d]c}} - \bar{\boldsymbol{\nabla}}_{[i} \bar{\boldsymbol{\nabla}}^c \frac{\delta S_{U,n-1}}{\delta \bar{\mathfrak{g}}^{d]c}}\right] \mathcal{F}^{id} + ST = 0 \tag{5.16}$$

Following the same reasoning used to derive (5.11), we obtain the identity

$$\bar{\boldsymbol{\nabla}}_{[i} \bar{\boldsymbol{\nabla}}^c \frac{\delta S_{U,n}}{\delta h^{d]c}} = \bar{\boldsymbol{\nabla}}_{[i} \bar{\boldsymbol{\nabla}}^c \frac{\delta S_{U,n-1}}{\delta \bar{\mathfrak{g}}^{d]c}} \tag{5.17}$$

If one recalls section 4.1, namely equation (4.17), one sees that this is equivalent to the self-coupling condition (5.18).

5.2 Iterative procedure III

In this chapter we intend to replicate section 4.2 for the TDiff case - see (4.20). Let us consider the equations corresponding to (4.21) and (4.22) (which belonged to the Diff case):

$$\frac{\delta U^{(0)}_{\text{int},h}}{\delta h^{ab}} = \left.\frac{\delta S_{U,h}}{\delta \gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}} \quad \text{and} \quad \frac{\delta U^{(n+1)}_{\text{int},h}}{\delta h^{ab}} = \left.\frac{\delta U^{(n)}_{\text{int},h}}{\delta \gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}} \tag{5.18}$$

$$\frac{\delta U^{(0)}_{\text{int},\varphi}}{\delta h^{ab}} = \left.\frac{\delta S_{U,\varphi}}{\delta \gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}} \quad \text{and} \quad \frac{\delta U^{(n+1)}_{\text{int},\varphi}}{\delta h^{ab}} = \left.\frac{\delta U^{(n)}_{\text{int},\varphi}}{\delta \gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}} \tag{5.19}$$

such that $S^{(n)}_{U,\text{int}} = U^{(n)}_{\text{int},h} + U^{(n)}_{\text{int},\varphi}$ $(n \geq 0)$. We then start with

$$\mathscr{A}_h = \frac{-1}{2} \int d^Dx \, K_{ab}{}^c{}_{ef}{}^d[\eta] \, \partial_c h^{ab} \partial_d h^{ef} \tag{5.20}$$

where

$$2K_{ab}{}^c{}_{ef}{}^d[\eta] = \eta^{cd}\eta_{a(e}\eta_{f)b} - b\eta^{cd}\eta_{ab}\eta_{ef} - 2\delta^c_{(e}\eta_{f)(a}\delta^d_{b)} + a\delta^c_{(e}\delta^d_{f)}\eta_{ab} + a\delta^c_{(a}\delta^d_{b)}\eta_{ef} \tag{5.21}$$

with $a \neq 1$ or $b \neq 1$. U-covariantization of (5.20) gives

$$S_{U,h} = \frac{-1}{2} \int d^D x \, \bar{\boldsymbol{\nabla}}_c h^{ab} \bar{\boldsymbol{\nabla}}_d h^{ef} K_{ab}{}^c{}_{ef}{}^d[\mathfrak{g}] \tag{5.22}$$

Let us write the self-coupling condition (5.18) denoting $U^{(N)}_{\text{int},h}$ by $S_{U,N+1}$ ($N \geq 0$):

$$\frac{\delta S_{U,1}}{\delta h^{ab}} = \left.\frac{\delta S_{U,h}}{\delta \gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}} \quad \text{and} \quad \frac{\delta S_{U,n+1}}{\delta h^{ab}} = \left.\frac{\delta S_{U,n}}{\delta \gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}} \tag{5.23}$$

where $n \geq 1$. Note that $\left.\frac{\delta S_{U,h/n}}{\delta \gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}}$, where $S_{U,h/n} = S_{U,h/n}[\bar{\mathfrak{g}}; h]$, stands for

$$\lim_{\gamma\to\bar{\mathfrak{g}}} \frac{\delta S_{U,h/n}[\gamma; h]}{\delta \gamma} + \lim_{\gamma\to\bar{\mathfrak{g}}} \frac{\delta S^{\text{NM}}_{U,0/n}[\gamma; h]}{\delta \gamma} \equiv \lim_{\gamma\to\bar{\mathfrak{g}}} \frac{\delta \tilde{S}_{U,0/n}}{\delta \gamma} \tag{5.24}$$

As we did in section 4.2, we start by assuming minimal coupling, such that

$$\tilde{S}_{U,0} = \frac{-1}{2} \int d^D x \, \nabla_c h^{ab} \nabla_d h^{ef} K_{ab}{}^c{}_{ef}{}^d \,, \tag{5.25}$$

In this section, if nothing is said, a capital latin letter stands for a sum of products of γ^{ab} and γ_{ab} multiplied by a function of $|\gamma|$ (if dependence on other metric is made explicit, the same applies with the components of that metric, its inverse and its determinant).

Recall that K obeys $K_{ab}{}^c{}_{ef}{}^d = K_{ef}{}^d{}_{ab}{}^c$ such that

$$\frac{\delta \tilde{S}_{U,0}}{\delta \gamma^{pq}} = \frac{-1}{2} A_{ab}{}^c{}_{ef}{}^d{}_{pq} \nabla_c h^{ab} \nabla_d h^{ef} - B_{pq}{}^c{}_{ef}{}^d{}_{ab} \nabla_c (h^{ab} \nabla_d h^{ef}) \tag{5.26}$$

where

$$\begin{aligned} A_{ab}{}^c{}_{ef}{}^d{}_{pq} &= \frac{\partial K_{ab}{}^c{}_{ef}{}^d}{\partial \gamma^{pq}} \quad \text{and} \\ B_{pq}{}^c{}_{ef}{}^d{}_{ab} &= \tilde{\Delta}^c{}_{pq\mu}{}^v{}_{(a} K_{b)v}{}^\mu{}_{ef}{}^d \end{aligned} \tag{5.27}$$

Thus we must have

$$\tilde{S}_{U,1} = \int d^D x \, X_{ab}{}^c{}_{ef}{}^d{}_{pq} h^{pq} \nabla_c h^{ab} \nabla_d h^{ef} \tag{5.28}$$

since the self-coupling condition implies that $\tilde{S}_{U,1}$ contains three h and two ∇ and any action term involving these will equal (5.28) up to ST.

$$\frac{\delta \tilde{S}_{U,1}}{\delta h^{pq}} = X_{ab}{}^c{}_{ef}{}^d{}_{pq} \nabla_c h^{ab} \nabla_d h^{ef} - 2 X_{pq}{}^c{}_{ef}{}^d{}_{ab} \nabla_c (h^{ab} \nabla_d h^{ef}) \tag{5.29}$$

such that

$$\left[\frac{\delta \tilde{S}_{U,1}}{\delta h^{pq}} = \frac{\delta \tilde{S}_{U,0}}{\delta \gamma^{pq}}\right]_{\gamma\to\bar{\mathfrak{g}}} \Rightarrow \begin{cases} X_{ab}{}^c{}_{ef}{}^d{}_{pq}[\bar{\mathfrak{g}}] = \frac{-1}{2} A_{ab}{}^c{}_{ef}{}^d{}_{pq}[\bar{\mathfrak{g}}] \\ B_{pq}{}^c{}_{ef}{}^d{}_{ab}[\bar{\mathfrak{g}}] + A_{pq}{}^c{}_{ef}{}^d{}_{ab}[\bar{\mathfrak{g}}] = 0 \end{cases} \tag{5.30}$$

Like in section 4.2, $B + A = 0$ is the consistency requirement in the first step of the iteration. Using (5.27), this is equivalent to

$$\left[\tilde{\Delta}^{c}{}_{pq\mu}{}^{\upsilon}{}_{(a}K_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d} + \frac{\partial K_{pq}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{ab}}\right]_{\gamma=\bar{\mathfrak{g}}} = 0 \tag{5.31}$$

Nonetheless, a general choice of parameters a and b in K doesn't solve the equation above. In section 4.2, the contribution of non-minimal couplings to the self-coupling conditions only affected the consistency requirements. Hence, the derivation of (4.53) and consequently (4.54) assuming minimal coupling was not in vain. The situation is different now, as we'll shortly see, such that it's worth introducing non-minimal couplings straightaway.

Recall that $\gamma \to \bar{\mathfrak{g}}$ is an abbreviation for $\mathcal{R}^{a}{}_{bcd}[\gamma] \to 0$ and $\partial_a|\gamma| \to 0$ which is equivalent to $|\gamma| \to |\bar{\mathfrak{g}}| = \varrho$ where ϱ is an undetermined constant. Since $\mathcal{R}^{a}{}_{bcd}$ is quadratic in ∇, we must use non-minimal coupling terms of the form

$$S^{\mathrm{NM}}_{U,0} = \frac{1}{2}\int d^D x \left(Q_i{}^{bcj}{}_{afed}\mathcal{R}^{i}{}_{bcj}h^{af}h^{ed} + \left[f(|\gamma|) - f(\varrho)\right] W_{ab}{}^{c}{}_{ef}{}^{d}\nabla_c h^{ab}\nabla_d h^{ef}\right) \tag{5.32}$$

(Q and W may include a function of $|\gamma|$ but this should not vanish when $\gamma \to \bar{\mathfrak{g}}$ or Q/W won't participate in the self-coupling condition).

$$\begin{aligned}
\frac{\delta S^{\mathrm{NM}}_{U,0}}{\delta\gamma^{pq}} &= \frac{1}{2}\frac{\partial Q_i{}^{bcj}{}_{afed}}{\partial\gamma^{pq}}\mathcal{R}^{i}{}_{bcj}h^{af}h^{ed} - Q_i{}^{\tau[dj]}{}_{abef}\tilde{\Delta}^{c}{}_{pq\tau}{}^{i}{}_{j}\nabla_c(h^{ab}\nabla_d h^{ef}) \\
&+ \left[f(|\gamma|) - f(\varrho)\right]\tilde{\Delta}^{c}{}_{pq\mu}{}^{\upsilon}{}_{(a}W_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d}\,\nabla_c(h^{ab}\nabla_d h^{ef}) + \frac{1}{2}\frac{\partial\left(\left[f(|\gamma|) - f(\varrho)\right] W_{ab}{}^{c}{}_{ef}{}^{d}\right)}{\partial\gamma^{pq}}\nabla_c h^{ab}\nabla_d h^{ef} \\
\Rightarrow \left.\frac{\delta S^{\mathrm{NM}}_{U,0}}{\delta\gamma^{pq}}\right|_{\gamma\to\bar{\mathfrak{g}}} &= -Q_i{}^{\tau[dj]}{}_{abef}[\bar{\mathfrak{g}}]\tilde{\Delta}^{c}{}_{pq\tau}{}^{i}{}_{j}[\bar{\mathfrak{g}}]\bar{\boldsymbol{\nabla}}_c(h^{ab}\bar{\boldsymbol{\nabla}}_d h^{ef}) + \frac{1}{2}\frac{\partial f(|\gamma|)}{\partial\gamma^{pq}}[\bar{\mathfrak{g}}]W_{ab}{}^{c}{}_{ef}{}^{d}[\bar{\mathfrak{g}}]\bar{\boldsymbol{\nabla}}_c h^{ab}\bar{\boldsymbol{\nabla}}_d h^{ef}
\end{aligned} \tag{5.33}$$

Now, in place of (5.25), $\tilde{S}_{U,0}$ is given by

$$\begin{aligned}
\tilde{S}_{U,0} = S^{\mathrm{NM}}_{U,0} - \frac{1}{2}\int d^D x\,\nabla_c h^{ab}\nabla_d h^{ef}K_{ab}{}^{c}{}_{ef}{}^{d} &= \frac{1}{2}\int d^D x\, Q_i{}^{bcj}{}_{afed}\mathcal{R}^{i}{}_{bcj}h^{af}h^{ed} \\
&- \frac{1}{2}\int d^D x\left[K_{ab}{}^{c}{}_{ef}{}^{d} - \left[f(|\gamma|) - f(\varrho)\right] W_{ab}{}^{c}{}_{ef}{}^{d}\right]\nabla_c h^{ab}\nabla_d h^{ef}
\end{aligned} \tag{5.34}$$

$$\begin{aligned}
\Rightarrow \left.\frac{\delta\tilde{S}_{U,0}}{\delta\gamma^{pq}}\right|_{\gamma\to\bar{\mathfrak{g}}} =& \frac{-1}{2}\left(A_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq} - \frac{\partial f(|\gamma|)}{\partial\gamma^{pq}}W_{ab}{}^{c}{}_{ef}{}^{d}\right)[\bar{\mathfrak{g}}]\bar{\boldsymbol{\nabla}}_c h^{ab}\bar{\boldsymbol{\nabla}}_d h^{ef} \\
&- \left(B_{pq}{}^{c}{}_{ef}{}^{d}{}_{ab} + Q_i{}^{\tau[dj]}{}_{abef}\tilde{\Delta}^{c}{}_{pq\tau}{}^{i}{}_{j}\right)[\bar{\mathfrak{g}}]\bar{\boldsymbol{\nabla}}_c(h^{ab}\bar{\boldsymbol{\nabla}}_d h^{ef})
\end{aligned} \tag{5.35}$$

While in section 4.2 the introduction of non-minimal coupling ammounted to substitute $B \to B + Q\tilde{\Delta}$ in the self-coupling condition (5.30), which only entered the consistency requirement, this time we also have

$$
\begin{aligned}
&A_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq} \to A_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq} - \frac{\partial f(|\gamma|)}{\partial \gamma^{pq}} W_{ab}{}^{c}{}_{ef}{}^{d} \\
\Leftrightarrow &A_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq} \to A_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq} + \frac{\sqrt{-|\gamma|}}{2} \frac{\partial f(|\gamma|)}{\partial \sqrt{-|\gamma|}} \gamma_{pq} W_{ab}{}^{c}{}_{ef}{}^{d}
\end{aligned}
\tag{5.36}
$$

which beyond affecting the consistency requirement also enters $\tilde{S}_{U,1}$.

Before proceding we're going to draw some inspiration from section 4.3, in particular the way its results related with the iterative procedure of section 4.2. Let's then bring back the assumption that $f(|\gamma|) \equiv f^{\omega}(|\gamma|) \propto \left(\sqrt{-|\gamma|}\right)^{\omega}$. Comparing $K^{\omega}{}_{ab}{}^{c}{}_{ef}{}^{d}[\gamma]$ from (4.93) with $K_{ab}{}^{c}{}_{ef}{}^{d}[\gamma]$ from (5.21), one sees that they are equal if $\omega = a$ and $b = 2a - 1$. What about choices of parameters where $b \neq 2a - 1$, like the case of *WTDiff symmetry* from [3]? Is there no solution to the iterative procedure? This seems to be false since Alvarez brings up a non-linear completion of the WTDiff lagrangian, namely action (56) that differs from the theories considered in section 4.3 by a term proportional to $\partial^{a} \ln|g| \ \partial_{a} \ln|g|$. As is known ([39]), it's possible to gauge fix the linear WTDiff theory and obtain the theory used in [10] of a transverse *spin-2 field* (see lagrangian (10)). Hence, we focus on parameters $a = \omega$ and $b = 2\omega - 1$ and replace K for K^{ω}.

Based on the aforementioned inspiration, we choose (5.32) with

$$
W_{ab}{}^{c}{}_{ef}{}^{d} = -\frac{K^{\omega}{}_{ab}{}^{c}{}_{ef}{}^{d}}{f^{\omega}(\varrho)}
\tag{5.37}
$$

and Q such that (recalling (4.99))

$$
\frac{1}{2} \int d^{D}x \, Q_{i}{}^{bcj}{}_{afed} \mathcal{R}^{i}{}_{bcj} h^{af} h^{ed} = S^{(2)}_{U,\mathcal{R}}
\tag{5.38}
$$

Hence, using (5.34), we have

$$
\tilde{S}_{U,0} = \frac{-1}{2 f^{\omega}(\varrho)} \int d^{D}x \, f^{\omega}(|\gamma|) K^{\omega}{}_{ab}{}^{c}{}_{ef}{}^{d} \nabla_{c} h^{ab} \nabla_{d} h^{ef} + \frac{1}{2} \int d^{D}x \, Q_{i}{}^{bcj}{}_{afed} \mathcal{R}^{i}{}_{bcj} h^{af} h^{ed}
\tag{5.39}
$$

and the self-coupling condition is

$$
\left[\frac{\delta \tilde{S}_{U,1}}{\delta h^{pq}} = \frac{\delta \tilde{S}_{U,0}}{\delta \gamma^{pq}}\right]_{\gamma \to \bar{\mathfrak{g}}} \Rightarrow \begin{cases} X_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq}[\bar{\mathfrak{g}}] = \frac{-1}{2} \left(\frac{\partial K^{\omega}{}_{ab}{}^{c}{}_{ef}{}^{d}}{\partial \gamma^{pq}}[\bar{\mathfrak{g}}] - \frac{\omega}{2} \frac{f^{\omega}(|\bar{\mathfrak{g}}|)}{f^{\omega}(\varrho)} \bar{\mathfrak{g}}_{pq} K^{\omega}{}_{ab}{}^{c}{}_{ef}{}^{d}[\bar{\mathfrak{g}}] \right) \\ \left[\tilde{\Delta}^{c}{}_{pq\mu}{}^{\upsilon}{}_{(a} K^{\omega}{}_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d} + Q_{i}{}^{\tau[dj]}{}_{abef} \tilde{\Delta}^{c}{}_{pq\tau}{}^{i}{}_{j} + \frac{\partial K^{\omega}{}_{ab}{}^{c}{}_{ef}{}^{d}}{\partial \gamma^{pq}} \right]_{\gamma = \bar{\mathfrak{g}}} = 0 \end{cases}
\tag{5.40}
$$

such that

$$S_{U,1} = \frac{-1}{2}\int d^Dx \left(\frac{\partial K^{\omega}{}_{ab}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{pq}}[\bar{\mathfrak{g}}] - \frac{\omega}{2}\bar{\mathfrak{g}}_{pq}K^{\omega}{}_{ab}{}^{c}{}_{ef}{}^{d}[\bar{\mathfrak{g}}]\right) h^{pq}\bar{\boldsymbol{\nabla}}_c h^{ab}\bar{\boldsymbol{\nabla}}_d h^{ef} \tag{5.41}$$

Had we assumed minimal coupling, the next step of the iterative procedure would start with

$$\begin{aligned} S_{U,1}[\gamma;h] &= \frac{-1}{2}\int d^Dx \left(\frac{\partial K^{\omega}{}_{ab}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{pq}} - \frac{\omega}{2}\gamma_{pq}K^{\omega}{}_{ab}{}^{c}{}_{ef}{}^{d}\right) h^{pq}\nabla_c h^{ab}\nabla_d h^{ef} \\ &= \frac{-1}{2}\int d^Dx\, \frac{1}{f^{\omega}(|\gamma|)}\frac{\partial f^{\omega}(|\gamma|)K^{\omega}{}_{ab}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{pq}} h^{pq}\nabla_c h^{ab}\nabla_d h^{ef} \\ &\equiv \frac{-1}{2}\int d^Dx\, \frac{A'{}_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq}}{f^{\omega}(|\gamma|)} h^{pq}\nabla_c h^{ab}\nabla_d h^{ef} \end{aligned} \tag{5.42}$$

However this is not the case and, following (5.37) and (5.38), we choose a non-minimal coupling for the second step such that

$$\begin{aligned} \tilde{S}_{U,1} = & \frac{-1}{2}\int d^Dx \left[\frac{A'{}_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq}}{f^{\omega}(|\gamma|)} - \left[f^{\omega}(|\gamma|) - f^{\omega}(\varrho)\right] W_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq}\right] h^{pq}\nabla_c h^{ab}\nabla_d h^{ef} + S^{(3)}_{U,\mathcal{R}} \\ = & \frac{-1}{2f^{\omega}(\varrho)}\int d^Dx\, A'{}_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq} h^{pq}\nabla_c h^{ab}\nabla_d h^{ef} + \int d^Dx\, \mathcal{R}^{i}{}_{\tau dj}Q_{i}{}^{\tau dj}{}_{abefst}h^{st}h^{ab}h^{ef} \end{aligned} \tag{5.43}$$

Hence

$$\frac{\delta\tilde{S}_{U,1}}{\delta\gamma^{pq}} = \frac{-1}{2}C_{ab}{}^{c}{}_{ef}{}^{d}{}_{ijpq}h^{ij}\nabla_c h^{ab}\nabla_d h^{ef} - D_{pq}{}^{c}{}_{ef}{}^{d}{}_{ijab}\nabla_c(h^{st}h^{ab}\nabla_d h^{ef}) \tag{5.44}$$

(ignoring terms proportional to the Riemann like in section 4.2) where

$$\begin{aligned} C_{ab}{}^{c}{}_{ef}{}^{d}{}_{ijpq} &= \frac{1}{f^{\omega}(\varrho)}\frac{\partial A'{}_{ab}{}^{c}{}_{ef}{}^{d}{}_{ij}}{\partial\gamma^{pq}} = \frac{1}{f^{\omega}(\varrho)}\frac{\partial^2 f^{\omega}(|\gamma|)K_{ab}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{pq}\partial\gamma^{ij}} \\ D_{pq}{}^{c}{}_{ef}{}^{d}{}_{stab} &= \frac{1}{2f^{\omega}(\varrho)}\tilde{\Delta}^{c}{}_{pq\mu}{}^{\upsilon}{}_{(a}A'{}_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d}{}_{st} - 3Q_{i}{}^{\tau[dj]}{}_{abefst}\tilde{\Delta}^{c}{}_{pq\tau}{}^{i}{}_{j} + (ab) \leftrightarrow (st) \end{aligned} \tag{5.45}$$

Up to ST, any action term involving four h and two ∇ will equal (5.46).

$$\tilde{S}_{U,2} = \int d^Dx\, Y_{ab}{}^{c}{}_{ef}{}^{d}{}_{ijpq}h^{pq}h^{ij}\nabla_c h^{ab}\nabla_d h^{ef} \tag{5.46}$$

$$\Rightarrow \frac{\delta\tilde{S}_{U,2}}{\delta h^{pq}} = 2Y_{ab}{}^{c}{}_{ef}{}^{d}{}_{ijpq}h^{ij}\nabla_c h^{ab}\nabla_d h^{ef} - 2Y_{pq}{}^{c}{}_{ef}{}^{d}{}_{ijab}\nabla_c(h^{ij}h^{ab}\nabla_d h^{ef}) \tag{5.47}$$

The self-coupling condition is

$$\left[\frac{\delta\tilde{S}_{U,2}}{\delta h^{pq}} = \frac{\delta\tilde{S}_{U,1}}{\delta\gamma^{pq}}\right]_{\gamma\to\bar{\mathfrak{g}}} \Rightarrow \begin{cases} 2Y_{ab}{}^{c}{}_{ef}{}^{d}{}_{ijpq}[\bar{\mathfrak{g}}] = \frac{-1}{2}C_{ab}{}^{c}{}_{ef}{}^{d}{}_{ijpq}[\bar{\mathfrak{g}}] \\ 2D_{pq}{}^{c}{}_{ef}{}^{d}{}_{ijab}[\bar{\mathfrak{g}}] + C_{pq}{}^{c}{}_{ef}{}^{d}{}_{ijab}[\bar{\mathfrak{g}}] = 0 \end{cases} \tag{5.48}$$

Based on the first and second step of the procedure, we postulate:

$$S_{U,n} = \frac{-1}{2f^\omega(\varrho)} \int \frac{d^Dx}{n!} \frac{\partial^n f^\omega(|\gamma|) K_{ab}{}^c{}_{ef}{}^d}{\partial\gamma^{kl}\dots\partial\gamma^{ij}}\bigg|_{\gamma\to\bar{\mathfrak{g}}} \overbrace{h^{ij}\dots h^{kl}}^{n \text{ times}} \bar{\boldsymbol{\nabla}}_c h^{ab} \bar{\boldsymbol{\nabla}}_d h^{ef} \tag{5.49}$$

($S_{U,0}$, $S_{U,1}$ and $S_{U,2}$ match the previously obtained expressions). Note that, since

$$\frac{\partial^n f^\omega(|\gamma|) K^\omega{}_{ab}{}^c{}_{ef}{}^d}{\partial\gamma^{kl}\dots\partial\gamma^{ij}}(x) = \int d^Dx_1\dots d^Dx_n \frac{\delta^n f^\omega(|\gamma|) K^\omega{}_{ab}{}^c{}_{ef}{}^d(x)}{\delta\gamma^{kl}(x_1)\dots\delta\gamma^{ij}(x_n)}\ , \tag{5.50}$$

we have

$$S_{U,n} = \frac{-1}{2f^\omega(\varrho)} \int \frac{d^Dx}{n!}\, d^Dx_1\dots d^Dx_n \frac{\delta^n f^\omega(|\gamma|) K^\omega{}_{ab}{}^c{}_{ef}{}^d(x)}{\delta\gamma^{kl}(x_1)\dots\delta\gamma^{ij}(x_n)}\bigg|_{\gamma\to\bar{\mathfrak{g}}} h^{ij}(x_1)\dots h^{kl}(x_n) \bar{\boldsymbol{\nabla}}_c h^{ab}(x) \bar{\boldsymbol{\nabla}}_d h^{ef}(x) \tag{5.51}$$

such that

$$\begin{aligned}
\sum_{n=0}^\infty \chi^n S_{U,n} &= \frac{-1}{2f^\omega(\varrho)} \int d^Dx \left(f^\omega(|\gamma|) K^\omega{}_{ab}{}^c{}_{ef}{}^d\right)[\bar{\mathfrak{g}}^{ab} + \chi h^{ab}] \bar{\boldsymbol{\nabla}}_c h^{ab} \bar{\boldsymbol{\nabla}}_d h^{ef} \\
&= \frac{-\chi^{-2}}{2f^\omega(\varrho)} \int d^Dx\, f^\omega(|g|) K^\omega{}_{ab}{}^c{}_{ef}{}^d[g] \bar{\boldsymbol{\nabla}}_c g^{ab} \bar{\boldsymbol{\nabla}}_d g^{ef} \\
&= S_{U,h} + \sum_{n=0}^\infty \chi^{n+1} U^{(n)}_{\mathrm{int},h} \equiv S_{U,h} + \chi U_{\mathrm{int},h}
\end{aligned} \tag{5.52}$$

where $g^{ab} = \bar{\mathfrak{g}}^{ab} + \chi h^{ab}$. You may be thinking that, like in section 4.2, starting with an action $\mathscr{A}_h$ we were led to an unique interacting theory but this is not the case here. The uniqueness of the result (5.52) of the iterative procedure is actually due to our assumption that $f(|\gamma|) = f^\omega(|\gamma|)$. To understand how can this be let us revisit section 4.3 to consider the non Einstein-Hilbert case, where $\omega \neq 1$ and we assume $\check{\Gamma}^b_{ba} = 0 \Leftrightarrow \partial_a\sqrt{-|\breve{g}|} = 0$ implying that $|\breve{g}|$ is a constant. Expanding $\mathfrak{L}_\omega$ around a metric $\breve{g}$ compatible with $\check{\nabla}$,

$$\mathfrak{L}_\omega|_{g^{ab}=\breve{g}^{ab}+\chi h^{ab}} = \frac{\chi^2 f^\omega(|\breve{g}|)}{2} K^\omega{}_{ab}{}^c{}_{ef}{}^d[\breve{g}] \check{\nabla}_c h^{ab} \check{\nabla}_d h^{ef} + O(\chi^3) \tag{5.53}$$

What if instead of $f^\omega(|g|)$ we had a general $f(|g|) = \sum_\omega f^\omega(|g|)$? Then, instead of $\mathfrak{L}_\omega$ in (4.86) we would have $\sum_\omega \mathfrak{L}_\omega$. Let us write $\sum_\omega \mathfrak{L}_\omega \equiv \sum_\omega f^\omega(|g|)\mathfrak{L}'_\omega$. This would equal $f(|g|)\mathfrak{L}'_{\bar{\omega}}$ if one was allowed to replace $\bar{\omega}$ by

$$\frac{\sum_\omega \omega f^\omega(|g|)}{f(|g|)} \equiv \bar{\omega}(|g|) \tag{5.54}$$

This is not in general a density weight but $\bar{\omega}(|\breve{g}|)$ is, since $|\breve{g}|$ is a constant. Hence one can write,

$$\begin{aligned}&\textstyle\sum_\omega f^\omega(|\breve{g}|)\mathfrak{L}'_\omega\big|_{g^{ab}=\breve{g}^{ab}+\chi h^{ab}} = f(|\breve{g}|)\mathfrak{L}'_{\bar{\omega}(|\breve{g}|)}\Big|_{g^{ab}=\breve{g}^{ab}+\chi h^{ab}} + O(\chi^3) \Leftrightarrow \\ &\textstyle\sum_\omega f^\omega(|\breve{g}|)\frac{\chi^2}{2}K^\omega{}_{ab}{}^c{}_{ef}{}^d[\breve{g}]\breve{\nabla}_c h^{ab}\breve{\nabla}_d h^{ef} = f(|\breve{g}|)\frac{\chi^2}{2}K^{\bar{\omega}(|\breve{g}|)}{}_{ab}{}^c{}_{ef}{}^d[\breve{g}]\breve{\nabla}_c h^{ab}\breve{\nabla}_d h^{ef} + O(\chi^3)\end{aligned} \tag{5.55}$$

where the l.h.s. is the lowest order term in $(\sum_\omega \mathfrak{L}_\omega)$'s expansion. Note that the r.h.s. is equally obtained starting with any $f(|g|) = \sum_\omega f^\omega(|g|)$ that leads to the same $f(|\breve{g}|)$ and $\bar{\omega}(|\breve{g}|)$ value. Hence, the iterative procedure requires choosing $f(|g|)$ in the first step, which is equivalent to choosing an infinite number of constants for each ω (a decision that is only constrained by two real numbers, $f(|\breve{g}|)$ and $\bar{\omega}(|\breve{g}|)$) and each choice leads to an a priori different theory.

Recall from the end of section 2.2, that a term

$$\alpha \int d^D x\, h^2 \equiv \mathscr{A}^*_h \tag{5.56}$$

(where α is a constant) can be added to (5.20) without breaking TDiff gauge invariance. U-covariantization of $\mathscr{A}^*_h$ gives

$$S^*_{U,h} = \alpha \int d^D x \left(h^{ab}\bar{\mathfrak{g}}_{ab}\right)^2 \tag{5.57}$$

Instead of inserting $\mathscr{A}^*_h$ into $\mathscr{A}_h$ and carrying out the iterations again, one can add

$$\frac{\delta S^*_{U,1}}{\delta h^{ab}} = \frac{\delta S^*_{U,h}}{\delta \gamma^{ab}}\bigg|_{\gamma\to\bar{\mathfrak{g}}} \quad \text{and} \quad \frac{\delta S^*_{U,n+1}}{\delta h^{ab}} = \frac{\delta S^*_{U,n}}{\delta \gamma^{ab}}\bigg|_{\gamma\to\bar{\mathfrak{g}}} \tag{5.58}$$

(where $n \geq 1$) to (5.18) and (5.19) such that now $S^{(N)}_{U,\text{int}} = U^{(N)}_{\text{int},h} + U^{(N)}_{\text{int},\varphi} + S^*_{U,N+1}$ $(N \geq 0)$. This comes from a reasoning similar to the one behind dividing (4.20) into the aforementioned equations (see the end of section 4.1): while $U^{(N)}_{\text{int},h}$ is quadratic in $\bar{\boldsymbol{\nabla}}$, $S^*_{U,N}$ is quadratic in the field h (hence, we're not loosing any solution of the iterative procedure because "it's impossible for any cancelation to occur between terms coming from" (5.18) and (5.58)). Also from this, one deduces that non-minimal coupling terms proportional to the Riemann (like the first in (5.32)) will not play a part in the following.

$S^*_{U,h}$ is of the form $\int d^D x\, h^{ab}h^{cd}E_{abcd}[\bar{\mathfrak{g}}]$ with $E_{abcd}[\bar{\mathfrak{g}}] = \alpha\, \bar{\mathfrak{g}}_{ab}\bar{\mathfrak{g}}_{cd}$ and the self-coupling condition is

$$\frac{\delta S^*_{U,1}}{\delta h^{ij}} = \frac{\delta S^*_{U,h}}{\delta \gamma^{ij}}\bigg|_{\gamma\to\bar{\mathfrak{g}}} = h^{ab}h^{cd}\frac{\partial E_{abcd}[\gamma]}{\partial \gamma^{ij}}\bigg|_{\gamma\to\bar{\mathfrak{g}}} + \frac{\delta S^{*\text{NM}}_{U,h}}{\delta \gamma^{ij}}\bigg|_{\gamma\to\bar{\mathfrak{g}}} \tag{5.59}$$

What if this is only satisfied with E as general as possible, i.e. $E_{abcd}[\bar{\mathfrak{g}}] = \alpha\, \bar{\mathfrak{g}}_{ab}\bar{\mathfrak{g}}_{cd} + \beta\, \bar{\mathfrak{g}}_{a(c}\bar{\mathfrak{g}}_{d)b}$ (mantaining $E_{abcd} = E_{bacd}$ and $E_{abcd} = E_{cdab}$ symmetries)? Then the gauge invariance of

$\mathscr{A}_h + \mathscr{A}_h^*$ is broken since the parameter β turns on mass term (2.33). Let us hope that (5.59) can be satisfied with $\beta = 0$ because we don't want to introduce extra DOFs by making our theory interacting as already mentioned in section 2.3.

$$S_{U,h}^{*\text{NM}} = \int d^D x \left(k \left[f^{\omega'}(|\gamma|) - f^{\omega'}(\rho) \right] \left(h^{ab}\gamma_{ab} \right)^2 + m_U^2 \left[f^{\omega}(|\gamma|) - f^{\omega}(\rho) \right] h^{ab}\gamma_{bc}h^{cd}\gamma_{da} \right) \tag{5.60}$$

where k and m_U^2 are constants, such that

$$\tilde{S}_{U,0}^{*} = \int d^D x \left[k f^{\omega'}(|\gamma|) + k_1 \right] \left(h^{ab}\gamma_{ab} \right)^2 + \int d^D x \left[m_U^2 f^{\omega}(|\gamma|) + k_2 \right] h^{ab}\gamma_{bc}h^{cd}\gamma_{da} \tag{5.61}$$

where $k_1 = \alpha - k f^{\omega'}(\rho)$ and $k_2 = \beta - m_U^2 f^{\omega}(\rho)$. Therefore, we have

$$\begin{aligned} \frac{\delta \tilde{S}_{U,0}^{*}}{\delta \gamma^{ab}} =& - \frac{\omega'}{2} k f^{\omega'}(|\gamma|) h^2 \gamma_{ab} - 2 \left[k f^{\omega'}(|\gamma|) + k_1 \right] h_{ab} h \\ & - \frac{\omega}{2} m_U^2 f^{\omega}(|\gamma|) h_{st} h^{st} \gamma_{ab} - 2 \left[m_U^2 f^{\omega}(|\gamma|) + k_2 \right] h_{as} \gamma^{st} h_{tb} \end{aligned} \tag{5.62}$$

$$\begin{aligned} \Rightarrow \left. \frac{\delta \tilde{S}_{U,0}^{*}}{\delta \gamma^{ab}} \right|_{\gamma \to \bar{\mathfrak{g}}} =& - \frac{\omega'}{2} k f^{\omega'}(|\bar{\mathfrak{g}}|) h^2 \bar{\mathfrak{g}}_{ab} - 2\alpha h_{ab} h \\ & - \frac{\omega}{2} m_U^2 f^{\omega}(|\bar{\mathfrak{g}}|) h_{st} h^{st} \bar{\mathfrak{g}}_{ab} - 2\beta h_{as} \bar{\mathfrak{g}}^{st} h_{tb} \end{aligned} \tag{5.63}$$

There's a solution for the self-coupling condition only if

$$\frac{\omega}{2} m_U^2 f^{\omega}(|\bar{\mathfrak{g}}|) = \alpha \tag{5.64}$$

and that is

$$\begin{aligned} S_{U,1}^{*} &= \int d^D x \left(-\frac{\omega'}{6} k f^{\omega'}(|\bar{\mathfrak{g}}|) h^3 - \frac{2\beta}{3} h_t^a h_b^t h_a^b - \frac{\omega}{2} m_U^2 f^{\omega}(|\bar{\mathfrak{g}}|) h_{st} h^{st} h \right) \\ &= \int d^D x \left(-\frac{\omega'}{6} k f^{\omega'}(|\bar{\mathfrak{g}}|) h^3 - \frac{2\beta}{3} h_t^a h_b^t h_a^b - \alpha h_{st} h^{st} h \right) \end{aligned} \tag{5.65}$$

As you can see, the solution depends on three independent parameters - $\omega' k f^{\omega'}(|\bar{\mathfrak{g}}|)$, β and α - and there's no issue in β being zero. Doing this, we've ended up with a 2-parameter family of solutions, so this time even assuming $f(|\gamma|) = f^{\omega}(|\gamma|)$ we would get an infinite number of interacting theories (that would get larger and larger as further parameters are introduced in each step of the procedure).

Let us end this section by pointing out that (5.64) comes from the fact that

$$\frac{\partial \left(h_{st} h^{st} h \right)}{\partial h^{ab}} = 2 h_{ab} h + h_{st} h^{st} \gamma_{ab} \tag{5.66}$$

and, due to the way it emerges in the iterative procedure, we'll also call this type of equations "consistency requirements".

5.3 Mass in the Diff case

Prompted by last section's final part where the possibility of massive theories was considered, we shift our focus again to the Diff case and consider $\mathscr{A}_h^{\mathrm{m}}$ given by (2.32). Recall from section 3.2 that its covariantization is denoted by S_h^{m}. The reader might be asking about section 4.1 in which we suggested that solving (4.16) automatically led to an interacting theory where the field h is massless. Actually, this reasoning depends on the fact that we solved it iteratively and started (4.18) with $\chi^{-1}S_{\text{int}}^{(-1)}$: we could have started with $\chi^{-n}S_{\text{int}}^{(-n)}$ such that $\frac{\delta S_{\text{int}}^{(-n)}}{\delta h^{ab}}=0$; indeed, this corresponds to having Cosmological Constant term - $\int d^Dx\,\Lambda\sqrt{-|\bar{g}|}$ - in S_h but this also wouldn't give us a massive theory since gauge invariance wouldn't be broken (assuming non-minimal coupling). Hence, to self-couple the mass terms like we're doing in the remainder of this work, we ignore the first term in (4.16).

In the same way (5.58) joined (5.18) and (5.19), we add ($n\geq 1$)

$$\frac{\delta S_1^{\mathrm{m}}}{\delta h^{ab}}=\left.\frac{\delta S_h^{\mathrm{m}}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}}\qquad\text{and}\qquad\frac{\delta S_{n+1}^{\mathrm{m}}}{\delta h^{ab}}=\left.\frac{\delta S_n^{\mathrm{m}}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{g}}\tag{5.67}$$

to (4.21) and (4.22) such that now $S_{\text{int}}^{(N)}=S_{\text{int},h}^{(N)}+S_{\text{int},\varphi}^{(N)}+S_{N+1}^{\mathrm{m}}$ ($N\geq 0$). Note that $\left.\frac{\delta S_{h/n}^{\mathrm{m}}}{\delta\gamma^{ab}}\right|_{\gamma\to\bar{\mathfrak{g}}}$, where $S_{h/n}^{\mathrm{m}}=S_{h/n}^{\mathrm{m}}[\bar{\mathfrak{g}};h]$, stands for

$$\lim_{\gamma\to\bar{\mathfrak{g}}}\frac{\delta S_{h/n}^{\mathrm{m}}[\gamma;h]}{\delta\gamma}+\lim_{\gamma\to\bar{\mathfrak{g}}}\frac{\delta S_{0/n}^{\mathrm{NM}}[\gamma;h]}{\delta\gamma}\equiv\lim_{\gamma\to\bar{\mathfrak{g}}}\frac{\delta\tilde{S}_{0/n}^{\mathrm{m}}}{\delta\gamma}\tag{5.68}$$

However, since the Riemman $\mathcal{R}[\gamma]$ is quadratic in ∇, non-minimal couplings won't be needed here to satisfy the self-coupling condition (5.67). $\tilde{S}_0^{\mathrm{m}}$ can be obtained from (5.61) by writing $k\equiv m^2$ and $m_U^2\equiv m^2k'$ and setting $k_1=0=k_2$ and $\omega=1=\omega'$:

$$\tilde{S}_0^{\mathrm{m}}=m^2\int d^Dx\,\sqrt{-|\gamma|}\left[\left(h^{ab}\gamma_{ab}\right)^2+k'h^{ab}\gamma_{bc}h^{cd}\gamma_{da}\right]\,;\tag{5.69}$$

such that using (5.62) one sees that

$$\tilde{S}_1^{\mathrm{m}}=-\frac{m^2}{6}\int d^Dx\sqrt{-|\gamma|}\left(h^3+4k'h_t^ah_b^th_a^b+3k'h_{st}h^{st}h\right)\tag{5.70}$$

and (5.64) corresponds to

$$k'=2\tag{5.71}$$

Moving on to the second step of the iterative procedure, we have

$$\begin{aligned}\frac{\delta\tilde{S}_1^{\mathrm{m}}}{\delta\gamma^{ab}}=&\frac{m^2}{12}\sqrt{-|\gamma|}\Big[h^3\gamma_{ab}+6h_{ab}h^2+12k'h_{as}\gamma^{st}h_{tb}h\\&+3k'h_{st}h^{st}\gamma_{ab}h+24k'h_{ae}h^{ed}h_{db}+4k'h_i^eh_e^ch_c^i\gamma_{ab}+6k'h_{st}h^{st}h_{ab}\Big]\end{aligned}\tag{5.72}$$

Now we get two consistency requirements that are identically satisfied (one of them is equivalent to (5.71)) such that a solution of the self-coupling condition exists:

$$
\begin{aligned}
\tilde{S}_2^{\mathrm{m}} =& \frac{m^2}{48} \int d^D x \sqrt{-|\gamma|} \Big[h^4 + 16k' h_i^e h_e^c h_c^i h \\
&+ 12 h_{st} h^{st} h^2 + 24 k' h^{st} h_{se} h^{ed} h_{dt} + 6k' h_{st} h^{st} h_{ef} h^{ef} \Big]
\end{aligned} \tag{5.73}
$$

(In Appendix D, we go over one more functional differentiation). But will these iterations converge? It's easy to see that the answer is yes since $\tilde{S}_0^{\mathrm{m}}$, $\tilde{S}_1^{\mathrm{m}}$ and $\tilde{S}_2^{\mathrm{m}}$ correspond to the terms of order $O(h^2)$, $O(h^3)$ and $O(h^4)$, respectively, of $8m^2\sqrt{|\gamma + h|}$ when Taylor expanded about γ. In fact one could already expect this from the fact that a Cosmological Constant term is of zero order in derivatives and can be added to the Einstein-Hilbert action while mantaining background independence and invariance under GCTs.

6 Tetrad gravity

The action $\mathscr{A}_h + \mathscr{A}_h^{\mathrm{m}}$ in last section, with $k' = 2$, propagates a scalar with negative kinetic energy - the celebrated *Ostrogradsky ghost/instability* (see reviews [40] and [41] for more on this). Only $k' = -1$ avoids this. The approach we're taking in the rest of the text is inspired by the formulation of dRGT massive gravity , whose quadratic expansion about the reference metric gives the FP mass term, in terms of tetrads [42].

6.1 EMTs of tetrad theories

Instead of choosing a coordinated basis $\{\partial_a\}$ (dual to $\{dx^a\}$) for the tangent bundle of our spacetime manifold, we can choose a less restrictive local basis[22] given by $\bar{e}_A = \bar{e}^a{}_A \partial_a$ (dual to $\bar{e}^A = \bar{e}_a{}^A dx^a$, where $\bar{e}_a{}^A$ is $\bar{e}^a{}_A$'s inverse[23]). We further require the local basis to be orthonormal, which is equivalent to ask the metric to obey $\bar{g} = \eta_{AB}\bar{e}^A \otimes \bar{e}^B$. We now deal with spacetime and Lorentz indices at the same time. To avoid mixing them, Lorentz indices will be denoted by upper case latin letters. We lower and raise these with η_{AB} and its inverse η^{AB}, respectively (this is consistent with the notation for the inverse tetrad, as can be seen through (6.1)).

In place of the metric $\bar{g}$ of section 3.2, we choose to represent our background flat spacetime by a tetrad with components $\bar{e}^a{}_A(x)$. We keep using the covariant derivative $\bar{\nabla}$ of the Levi-Civita connection Γ built from $\bar{g}_{ab} = \eta_{AB}\bar{e}_a{}^A\bar{e}_b{}^B$. We can say we have a flat tetrad, meaning that $\mathcal{R}^a{}_{bcd}[\Gamma] = 0$ (like before, this is only used for the conservation of the canonical EMT; all other results are independent of the covariant derivatives commuting or not). Bear in mind that we're working with arbitrary non-inertial coordinates x^a (here, non-inertial means that $\bar{e}_a{}^A(x) \neq \delta_a^A$ implying $\bar{g}_{ab}(x) \neq \eta_{ab}$). Lastly, let us write the law of transformation of the tetrad under Local Lorentz Transformations (LLTs) for future reference:

$$\bar{e}^a{}_A(x) \to \Lambda_A{}^B(x)\bar{e}^a{}_B(x) \tag{6.2}$$

We now focus on field theories with a (1,0) tensor field and Lorentz covector with components $f^a{}_A(x)$ (besides an generic collection of dynamical fields φ like before) and ask these theories

[22] By "less restrictive", we're allowing non-coordinated basis such that we may have $d\bar{e}^a \neq 0 \Rightarrow \partial_{[a}\bar{e}_{b]}{}^A \neq 0$. In this basis,

$$\bar{g} = \bar{g}_{ab}dx^a \otimes dx^b = \bar{g}_{AB}\bar{e}^A \otimes \bar{e}^B \Rightarrow \bar{g}_{ab} = \bar{g}_{AB}\bar{e}_a{}^A\bar{e}_b{}^B \Leftrightarrow \bar{g}^{ab} = \bar{g}^{AB}\bar{e}^a{}_A\bar{e}^b{}_B \tag{6.1}$$

[23] $e_a{}^A e^b{}_A = \delta_a^b$ and $e_a{}^A e^a{}_B = \delta_B^A$.

to be Lorentz covariant in addition to the usual (general) covariance. The action must then be a scalar

$$S = \int d^D x\, \mathscr{L}[\bar{e}^a{}_A, \partial_b \bar{e}^a{}_A; f^a{}_A, \partial_b f^a{}_A, \varphi, \partial_b \varphi] \tag{6.3}$$

with all Lorentz indices contracted. (Note that this is not sufficient. Since $\bar{\nabla}_a \Lambda_A^B \neq 0$ we need a "gauge covariant derivative" $\mathfrak{D}_a$ in place of $\bar{\nabla}_a$ such that

$$\mathfrak{D}_a T^{ABC...}{}_{DEF...} \to \Lambda^A{}_{A'} \Lambda^B{}_{B'} \Lambda^C{}_{C'} ... \Lambda_D^{D'} \Lambda_E^{E'} \Lambda_F^{F'} ... \mathfrak{D}_a T^{A'B'C'...}{}_{D'E'F'...} \tag{6.4}$$

where T is some object transforming under LLT. We assume that the connection associated with the Lorentz group, "$\mathfrak{D}_a - \bar{\nabla}_a$", is made of $\bar{e}$ and $\bar{\nabla}$ alone such that we'll forget about $\mathfrak{D}$. The information that the lagrangian can be written in terms of $\mathfrak{D}$ and no other derivative is concealed in Lorentz invariance).

Like in section 3.3, we take into consideration the conclusion drawn in 3.2 and jump ahead to a choice of independent variables like (3.12):

$$\mathscr{L} \equiv \tilde{\mathscr{L}} = \tilde{\mathscr{L}}[\bar{e}^a{}_A, \bar{\nabla}_b \bar{e}^a{}_A; f^a{}_A, \bar{\nabla}_b f^a{}_A, \varphi, \bar{\nabla}_b \varphi] \tag{6.5}$$

$\bar{\nabla}_b \bar{e}^a{}_A$ was included as a variable since, even though metric compatibility fully determines $\bar{\nabla}\bar{g}$, the same doesn't happen with $\bar{\nabla}\bar{e}$ (see (6.8)). We write $\tilde{\mathscr{L}} \equiv |\bar{e}|\tilde{\mathcal{L}}$ (note that $|\bar{g}| = -|\bar{e}|^2$ where $|\bar{e}| \equiv det\left(\bar{e}_a{}^A\right)$) and, taking into account commutativity between variation and covariant derivative, one arrives at the EOMs:

$$\begin{cases} 0 = \frac{\partial \tilde{\mathscr{L}}}{\partial f^a{}_A} - \bar{\nabla}_b \left(\frac{\partial \tilde{\mathscr{L}}}{\partial \bar{\nabla}_b f^a{}_A} \right) = \frac{\partial \mathscr{L}}{\partial f^a{}_A} - \partial_b \left(\frac{\partial \mathscr{L}}{\partial \partial_b f^a{}_A} \right) \equiv \frac{\delta S}{\delta f^a{}_A} \\ 0 = \frac{\partial \tilde{\mathscr{L}}}{\partial \varphi} - \bar{\nabla}_b \left(\frac{\partial \tilde{\mathscr{L}}}{\partial \bar{\nabla}_b \varphi} \right) = \frac{\partial \mathscr{L}}{\partial \varphi} - \partial_b \left(\frac{\partial \mathscr{L}}{\partial \partial_b \varphi} \right) \equiv \frac{\delta S}{\delta \varphi} \end{cases} \tag{6.6}$$

Let us introduce further notation that will be useful:

$$\frac{\delta \tilde{\mathcal{S}}}{\delta \bar{e}^a{}_A} \equiv \left[\frac{\partial \tilde{\mathcal{L}}}{\partial \bar{e}^a{}_A} - \bar{\nabla}_b \left(\frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_b \bar{e}^a{}_A} \right) \right] \tag{6.7}$$

and the same with $f^a{}_A$ or φ in place of $\bar{e}^a{}_A$ such that $\frac{\delta S}{\delta f^a{}_A} \equiv |\bar{e}| \frac{\delta \tilde{\mathcal{S}}}{\delta f^a{}_A}$ and $\frac{\delta S}{\delta \varphi} \equiv |\bar{e}| \frac{\delta \tilde{\mathcal{S}}}{\delta \varphi}$. Before proceding, note that

$$\bar{\nabla}_b (\bar{e}^a{}_A \bar{e}_{aB}) = \bar{\nabla}_b \eta_{AB} = \partial_b \eta_{AB} = 0 \Rightarrow \bar{e}^a{}_A \bar{\nabla}_b \bar{e}_{aB} = \bar{e}_{aA} \bar{\nabla}_b \bar{e}^a{}_B = -\bar{e}_{aB} \bar{\nabla}_b \bar{e}^a{}_A \tag{6.8}$$

such that $\bar{e}_{aB} \bar{\nabla}_b \bar{e}^a{}_A \equiv \bar{\omega}_{bAB} = -\bar{\omega}_{bBA}$. Writing $f^a{}_A$ and φ as ϕ^σ with $\sigma = 1$ and 2, respectively, we have $\tilde{\mathscr{L}}[\bar{e}^a{}_A, \bar{\nabla}_b \bar{e}^a{}_A; f^a{}_A, \bar{\nabla}_b f^a{}_A, \varphi, \bar{\nabla}_b \varphi] \equiv \tilde{\mathscr{L}}[\bar{e}^a{}_A, \bar{\nabla}_b \bar{e}^a{}_A; \phi^\sigma, \bar{\nabla}_a \phi^\sigma]$.

$$\begin{aligned} \bar{\nabla}_b \tilde{\mathcal{L}} = \partial_b \tilde{\mathcal{L}} &= \frac{\partial \tilde{\mathcal{L}}}{\partial \phi^\sigma} \partial_b \phi^\sigma + \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_c \phi^\sigma} \partial_b \bar{\nabla}_c \phi^\sigma + \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{e}^a{}_A} \partial_b \bar{e}^a{}_A + \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_c \bar{e}^a{}_A} \partial_b \bar{\nabla}_c \bar{e}^a{}_A \\ &= \frac{\partial \tilde{\mathcal{L}}}{\partial \phi^\sigma} \bar{\nabla}_b \phi^\sigma + \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_c \phi^\sigma} \bar{\nabla}_b \bar{\nabla}_c \phi^\sigma + \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{e}^a{}_A} \bar{\nabla}_b \bar{e}^a{}_A + \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_c \bar{e}^a{}_A} \bar{\nabla}_b \bar{\nabla}_c \bar{e}^a{}_A \end{aligned} \tag{6.9}$$

Using (6.8) and $\bar{\nabla}$'s commutativity, the equation above is equivalent to

$$\bar{\nabla}_b \tilde{\mathscr{L}} = \frac{\partial \tilde{\mathscr{L}}}{\partial \phi^\sigma} \bar{\nabla}_b \phi^\sigma + \frac{\partial \tilde{\mathscr{L}}}{\partial \bar{\nabla}_c \phi^\sigma} \bar{\nabla}_b \bar{\nabla}_c \phi^\sigma + |\bar{e}| \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{e}^a{}_A} \bar{e}^{aB} \bar{\omega}_{bAB} + |\bar{e}| \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_c \bar{e}^a{}_A} \bar{\nabla}_c \bar{e}^{aB} \bar{\omega}_{bAB} \tag{6.10}$$

and leads (using $\bar{\nabla}$ commutativity again) to

$$\begin{aligned} \bar{\nabla}_d \left(\delta^d_b \tilde{\mathscr{L}} - \frac{\partial \tilde{\mathscr{L}}}{\partial \bar{\nabla}_d \phi^\sigma} \bar{\nabla}_b \phi^\sigma - |\bar{e}| \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_d \bar{e}^a{}_A} \bar{e}^{aB} \bar{\omega}_{bAB} \right) &= \frac{\delta S}{\delta \phi^\sigma} \bar{\nabla}_b \phi^\sigma + |\bar{e}| \frac{\partial \tilde{\mathcal{S}}}{\partial \bar{e}^a{}_A} \bar{e}^{aB} \bar{\omega}_{bAB} \\ \bar{\nabla}_d \left(\delta^d_b \tilde{\mathscr{L}} - \frac{\partial \tilde{\mathscr{L}}}{\partial \bar{\nabla}_d \phi^\sigma} \bar{\nabla}_b \phi^\sigma - |\bar{e}| \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_d \bar{e}^a{}_{[A}} \bar{e}^{aB]} \bar{\omega}_{bAB} \right) &= \frac{\delta S}{\delta \phi^\sigma} \bar{\nabla}_b \phi^\sigma + |\bar{e}| \frac{\partial \tilde{\mathcal{S}}}{\partial \bar{e}^a{}_{[A}} \bar{e}^{aB]} \bar{\omega}_{bAB} \end{aligned} \tag{6.11}$$

where we used $\bar{\omega}_{aAB}$ antisymmetry. Since $\Lambda_A{}^B = \delta_A^B + \epsilon_A{}^B + O(\epsilon^2)$, under infinitesimal LLT,

$$\begin{aligned} \hat{\delta} \bar{e}^a{}_A(x) &= \bar{e}^a{}_B(x) \epsilon^B{}_A(x) \\ \hat{\delta} f^a{}_A(x) &= f^a{}_B(x) \epsilon^B{}_A(x) \end{aligned} \tag{6.12}$$

where $\epsilon_{AB} = -\epsilon_{BA}$. Since $\tilde{\mathcal{L}}$ and Γ are Lorentz scalars, the variation of $\tilde{\mathcal{L}}$ under a LLT of all Lorentz indices but the ones inside Γ is null:

$$\begin{aligned} 0 = \hat{\delta} \tilde{\mathcal{L}} &= \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{e}^a{}_A} \hat{\delta} \bar{e}^a{}_A + \frac{\partial \tilde{\mathcal{L}}}{\partial f^a{}_A} \hat{\delta} f^a{}_A + \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_b \bar{e}^a{}_A} \bar{\nabla}_b \hat{\delta} \bar{e}^a{}_A + \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_b f^a{}_A} \bar{\nabla}_b \hat{\delta} f^a{}_A \\ &= \frac{\partial \tilde{\mathcal{S}}}{\partial \bar{e}^a{}_A} \hat{\delta} \bar{e}^a{}_A + \frac{\delta \tilde{\mathcal{S}}}{\delta f^a{}_A} \hat{\delta} f^a{}_A + \partial_b \left(\frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_b \bar{e}^a{}_A} \hat{\delta} \bar{e}^a{}_A + \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_b f^a{}_A} \hat{\delta} f^a{}_A \right) \\ = \left(\frac{\partial \tilde{\mathcal{S}}}{\partial \bar{e}^a{}_{[A}} \bar{e}^{aB]} + \frac{\delta \tilde{\mathcal{S}}}{\delta f^a{}_{[A}} f^{aB]} \right) \epsilon_{BA} &+ \partial_b \left[\left(\frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_b \bar{e}^a{}_{[A}} \bar{e}^{aB]} + \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_b f^a{}_{[A}} f^{aB]} \right) \epsilon_{BA} \right] \end{aligned} \tag{6.13}$$

Using arbitrariness of the transformation parameter ϵ_{AB} (since we are not transforming Γ, $\hat{\delta}$ and $\bar{\nabla}$ commute), one sees that both expressions inside rounded brackets above must vanish. This is equivalent to:

$$|\bar{e}| \frac{\partial \tilde{\mathcal{S}}}{\partial \bar{e}^a{}_{[A}} \bar{e}^{aB]} = -\frac{\delta S}{\delta f^a{}_{[A}} f^{aB]} \tag{6.14}$$

$$|\bar{e}| \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{\nabla}_b \bar{e}^a{}_{[A}} \bar{e}^{aB]} = -\frac{\partial \tilde{\mathscr{L}}}{\partial \bar{\nabla}_b f^a{}_{[A}} f^{aB]} \tag{6.15}$$

Using this on (6.11), we arrive at the on-shell covariant conservation law

$$\bar{\nabla}_d \left(\delta^d_b \tilde{\mathscr{L}} - \frac{\partial \tilde{\mathscr{L}}}{\partial \bar{\nabla}_d f^a{}_A} \left(\bar{\nabla}_b f^a{}_A - f^{aB} \bar{\omega}_{bAB} \right) - \frac{\partial \tilde{\mathscr{L}}}{\partial \bar{\nabla}_d \varphi} \bar{\nabla}_b \varphi \right) = \frac{\delta S}{\delta f^a{}_A} \left(\bar{\nabla}_b f^a{}_A - f^{aB} \bar{\omega}_{bAB} \right) + \frac{\delta S}{\delta \varphi} \bar{\nabla}_b \varphi \tag{6.16}$$

(we can drop antisymmetrization on A and B indices when they are contracted with $\bar{\omega}_{bAB}$) such that

$$
\begin{aligned}
T^{cd}_{\text{Can}} &\equiv \frac{1}{|\bar{e}|}\left(\frac{\partial\tilde{\mathscr{L}}}{\partial\bar{\nabla}_d f^a{}_A}\left(\bar{\nabla}_b f^a{}_A - f^{aB}\bar{\omega}_{bAB}\right) + \frac{\partial\tilde{\mathscr{L}}}{\partial\bar{\nabla}_d\varphi}\bar{\nabla}_b\varphi - \delta^d_b\tilde{\mathscr{L}}\right)\bar{e}^b{}_B\bar{e}^{cB} \\
&= \frac{1}{|\bar{e}|}\left(\frac{\partial\mathscr{L}}{\partial\partial_d f^a{}_A}\left(\bar{\nabla}_b f^a{}_A - f^{aB}\bar{\omega}_{bAB}\right) + \frac{\partial\mathscr{L}}{\partial\partial_d\varphi}\bar{\nabla}_b\varphi - \delta^d_b\mathscr{L}\right)\bar{e}^b{}_B\bar{e}^{cB}
\end{aligned}
\tag{6.17}
$$

We also derive Rosenfeld's EMT from $\mathscr{L}$ by taking the active transformation perpective on infinitesimal GCT and using gauge invariance associated with it. Like before, we compare $\delta\mathscr{L}$ caused by an arbitrary variation of $\bar{e}^a{}_A$, $f^a{}_A$ and φ with

$$\delta\mathscr{L} = \partial_a\left(\xi^a\mathscr{L}\right) \tag{3.22}$$

Then we enforce the EOMs and use $\delta\bar{e}^a{}_A$ for a infinitesimal GCT explicitly:

$$\delta\bar{e}^a{}_A = \xi^b\bar{\nabla}_b\bar{e}^a{}_A - \bar{e}^b{}_A\bar{\nabla}_b\xi^a = \xi^b\bar{\omega}_{bAB}\bar{e}^{aB} - \bar{e}^b{}_A\bar{\nabla}_b\xi^a \tag{6.18}$$

We have

$$\partial_a\left(\xi^a\mathscr{L}\right) = \frac{\delta S}{\delta f^a{}_A}\delta f^a{}_A + \frac{\delta S}{\delta\varphi}\delta\varphi + \frac{\delta S}{\delta\bar{e}^a{}_A}\delta\bar{e}^a{}_A + \partial_a(...)^a \tag{6.19}$$

where $\frac{\delta S}{\delta\bar{e}^a{}_A} \equiv \frac{\partial\mathscr{L}}{\partial\bar{e}^a{}_A} - \partial_c\left(\frac{\partial\mathscr{L}}{\partial\partial_c\bar{e}^a{}_A}\right)$. Using the EOMs (6.6), substituting (6.18), and integrating by parts, we arrive at

$$\int d^Dx\left[\bar{\omega}_{aAB}\frac{\delta S}{\delta\bar{e}^b{}_A}\bar{e}^{bB} + \bar{\nabla}_b\left(\frac{\delta S}{\delta\bar{e}^a{}_A}e^b{}_A\right)\right]\xi^a + ST = 0 \tag{6.20}$$

Since ∂ and $\hat{\delta}$ commute, under infinitesimal LLT

$$\hat{\delta}S = \int d^Dx\left(\frac{\delta S}{\delta\bar{e}^a{}_A}\hat{\delta}\bar{e}^a{}_A + \frac{\delta S}{\delta f^a{}_A}\hat{\delta}f^a{}_A\right) + ST = 0 \tag{6.21}$$

We can convert the surface term into an integral over the boundary and, using an appropriate choice of boundary conditions for the transformation parameter ϵ_{AB}, it can be made to vanish. Substituting (6.12), one has

$$\int d^Dx\left(\frac{\delta S}{\delta\bar{e}^a{}_{[A}}\bar{e}^{aB]} + \frac{\delta S}{\delta f^a{}_{[A}}f^{aB]}\right)\epsilon_{BA} = 0 \tag{6.22}$$

since $\epsilon_{AB} = -\epsilon_{BA}$. One then obtains, using arbitrariness of ϵ_{AB}, the off-shell identity

$$\frac{\delta S}{\delta\bar{e}^a{}_{[A}}\bar{e}^{aB]} = -\frac{\delta S}{\delta f^a{}_{[A}}f^{aB]} \tag{6.23}$$

Hence, since $\bar{\omega}_{aAB}$ is antisymmetric on A and B, the first term in (6.20) is proportional to f's EOM, vanishing on-shell. Again the surface term is made to vanish, throuh an appropriate choice of boundary conditions for ξ^a. Due to its arbitrariness, we have the following on-shell covariant conservation law:

$$\bar{\nabla}_b \left(\frac{\delta S}{\delta \bar{e}^a{}_A} \bar{e}^b{}_A \right) = 0 \Leftrightarrow \bar{\nabla}_b \left(\frac{\delta S}{\delta \bar{e}_b{}^A} \bar{e}^{aA} \right) = 0 \tag{6.24}$$

(Note that $\frac{\delta S}{\delta \bar{e}^a{}_A} = -\bar{e}_b{}^A \bar{e}_a{}^B \frac{\delta S}{\delta \bar{e}_b{}^B}$). The Rosenfeld's EMT (which like the canonical one could be defined with an extra off-shell divergenceless term) is

$$T^{ab}_{\text{Ros}} \equiv \frac{-1}{|\bar{e}|} \frac{\delta S}{\delta \bar{e}_b{}^A} \bar{e}^{aA} = \frac{1}{|\bar{e}|} \frac{\delta S}{\delta \bar{e}^c{}_A} \bar{e}^b{}_A \bar{g}^{ca} \,. \tag{6.25}$$

This is not new, but we still call attention to the fact that conservation law (6.24) is valid independently of the metric being flat or curved. Since it has become customary, we now derive the relation between the canonical and Rosenfeld's EMTs of "tetrad theories". Integrating (6.19), one obtains

$$\int d^D x \left(\frac{\delta S}{\delta f^a{}_A} \delta f^a{}_A + \frac{\delta S}{\delta \varphi} \delta \varphi + \frac{\delta S}{\delta \bar{e}^a{}_A} \delta \bar{e}^a{}_A \right) + ST = 0 \tag{6.26}$$

In addition to (6.18) and (3.31) (we stick with φ being, for convenience, a scalar), we substitute

$$\delta f^a{}_A = \xi^b \bar{\nabla}_b f^a{}_A - f^b{}_A \bar{\nabla}_b \xi^a \tag{6.27}$$

in (6.26):

$$\begin{aligned} &\int d^D x \Big(- \frac{\delta S}{\delta f^a{}_A} f^b{}_A \bar{\nabla}_b \xi^a + \xi^b \frac{\delta S}{\delta f^a{}_A} \bar{\nabla}_b f^a{}_A + \xi^b \frac{\delta S}{\delta \varphi} \bar{\nabla}_b \varphi + \xi^b \frac{\delta S}{\delta \bar{e}^a{}_A} \bar{\omega}_{bAB} \bar{e}^{aB} - \frac{\delta S}{\delta \bar{e}^a{}_A} \bar{e}^b{}_A \bar{\nabla}_b \xi^a + ST \Big) \\ &\qquad = \int d^D x \Big(- \frac{\delta S}{\delta f^a{}_A} f^b{}_A \bar{\nabla}_b \xi^a - |\bar{e}| \xi^b \bar{\nabla}_d T^{cd}_{\text{Can}} \bar{e}_{cB} \bar{e}_b{}^B - \frac{\delta S}{\delta \bar{e}^a{}_A} \bar{e}^b{}_A \bar{\nabla}_b \xi^a + ST \Big) = 0 \end{aligned} \tag{6.28}$$

where we had to use (6.23) before inserting the canonical EMT. Integrating by parts,

$$\int d^D x \left[\bar{\nabla}_b \left(\frac{\delta S}{\delta f^a{}_A} f^b{}_A \right) - |\bar{e}| \bar{\nabla}_d T^{cd}_{\text{Can}} \bar{e}_{cB} \bar{e}_a{}^B + \bar{\nabla}_b \left(\frac{\delta S}{\delta \bar{e}^a{}_A} \bar{e}^b{}_A \right) \right] \xi^a + ST = 0 \tag{6.29}$$

Due to arbitrariness of the transformation parameter $\xi^a(x)$ one obtains the following off-shell identity that doesn't involve the transformation parameters:

$$-\bar{\nabla}_b \left(\frac{\delta S}{\delta \bar{e}^a{}_A} \bar{e}^b{}_A \right) = \frac{\delta S}{\delta \varphi} \bar{\nabla}_a \varphi + \frac{\delta S}{\delta f^b{}_A} \left(\bar{\nabla}_a f^b{}_A - f^{bB} \bar{\omega}_{aAB} \right) + \bar{\nabla}_b \left(\frac{\delta S}{\delta f^a{}_A} f^b{}_A \right) \tag{6.30}$$

where we have expanded T^{cd}_{Can} according to (6.16), as will turn useful.

6.2 Reverse engineering GR in terms of tetrads

Now, instead of a metric theory of gravity, we consider a "tetrad theory" such that $S \equiv S[e]$. One can "divide" the components of the tetrad $e^a{}_A$ into a non-dynamical (flat) "background tetrad" $\bar{e}^a{}_A$ and a dynamical field $f^a{}_A$:

$$S[e]\Big|_{e^a{}_A=\phi(\chi f^a{}_A,\bar{e}^a{}_A)} \equiv S[\bar{e};f] = \sum_{n=0}^{\infty} \chi^n S^{(n)}[\bar{e};f] \tag{6.31}$$

We consider

$$S[e] \equiv \int d^D x\, K_a^b{}^{Ac}{}_d{}^D[e]\bar{\nabla}_b e^a{}_A \bar{\nabla}_c e^d{}_D \tag{6.32}$$

such that, assuming $\phi = \bar{e}^a{}_A + \chi f^a{}_A$:

$$\begin{aligned} S[\bar{e};f] &= \sum_{n=2}^{\infty} \frac{\chi^n}{n!} \int d^D x_1 ... \, d^D x_n \frac{\delta^n S[e]}{\delta e^b{}_B(x_1) ... \, \delta e^c{}_C(x_n)}\Big|_{e=\bar{e}} f^b{}_B(x_1) ... \, f^c{}_C(x_n) \\ &\equiv \sum_{n=0}^{\infty} \chi^n S_n[\bar{e};f] \; . \end{aligned} \tag{6.33}$$

The proof that $S^{(0)} = 0 = S^{(1)}$ is in appendix E. We assume that $S[e=\phi] = S[\bar{e};f]$ is a Lorentz and spacetime scalar, such that S_n are too. However, in first place, we are going to analyse the impact of the background independent action $S[e]$ being invariant under (infinitesimal) LLT. We have $\hat{\delta}_\epsilon S[e] = 0$ with $\hat{\delta}_\epsilon e^a{}_A = e^a{}_B \epsilon^B{}_A$, where $\epsilon_{AB} = -\epsilon_{BA}$. Note that

$$\hat{\delta}_\epsilon e^a{}_A\Big|_{e=\phi} = \bar{e}^a{}_B \epsilon^B{}_A + \chi f^a{}_B \epsilon^B{}_A$$

such that

$$\hat{\delta}_\epsilon S[\bar{e};f] = 0 \quad \text{with} \quad \hat{\delta}_\epsilon \bar{e}^a{}_A + \chi \hat{\delta}_\epsilon f^a{}_A = \bar{e}^a{}_B \epsilon^B{}_A + \chi f^a{}_B \epsilon^B{}_A \; .$$

We choose $\hat{\delta}_\epsilon \bar{e}^a{}_A = 0$ and $\hat{\delta}_\epsilon f^a{}_A = \chi^{-1} \hat{\delta}_\epsilon^{(-1)} f^a{}_A + \hat{\delta}_\epsilon^{(0)} f^a{}_A$ where

$$\begin{aligned} \hat{\delta}_\epsilon^{(-1)} f^a{}_A &= \bar{e}^a{}_B \epsilon^B{}_A \\ \hat{\delta}_\epsilon^{(0)} f^a{}_A &= f^a{}_B \epsilon^B{}_A \end{aligned} \tag{6.34}$$

Note that all functional integrals (except for the background independent $S[e]$) should be automatically understood to have $\bar{e}$ and f as variables.

$$\begin{aligned} \hat{\delta}_\epsilon S_n = \int d^D x \frac{\delta S_n}{\delta f^a{}_A} \hat{\delta}_\epsilon f^a{}_A + ST &= \int d^D x \left(\chi^{-1} \frac{\delta S_n}{\delta f^a{}_A} \hat{\delta}_\epsilon^{(-1)} f^a{}_A + \frac{\delta S_n}{\delta f^a{}_A} \hat{\delta}_\epsilon^{(0)} f^a{}_A \right) + ST \\ &\equiv \chi^{-1} \hat{\delta}_\epsilon^{(-1)} S_n + \hat{\delta}_\epsilon^{(0)} S_n \end{aligned} \tag{6.35}$$

Hence, since S is invariant under LLTs:

$$\hat{\delta}_\epsilon^{(-1)} S_0 + ST = 0 \tag{6.36}$$

$$\hat{\delta}_\epsilon^{(0)} S_{n-1} + \hat{\delta}_\epsilon^{(0)} S_n + ST = 0 \;, \quad n \geq 1 \tag{6.37}$$

Using (6.36), one has that

$$\hat{\delta}_\epsilon^{(-1)} S_0 = \int d^D x \, \frac{\delta S_0}{\delta f^a{}_A} \bar{e}^a{}_B \epsilon^B{}_A + ST = \int d^D x \, \frac{\delta S_0}{\delta f^a{}_{[A}} \bar{e}^{aB]} \epsilon_{BA} + ST = 0 \tag{6.38}$$

We convert the surface term into an integral over the boundary such that it vanishes using an suitable choice of boundary conditions for the parameter ϵ_{AB}. One then arrives at the identity

$$\frac{\delta S_0}{\delta f^a{}_{[A}} \bar{e}^{aB]} = 0 \Leftrightarrow \frac{\delta S_0}{\delta f^{[a}{}_A} \bar{e}_{b]A} = 0 \tag{6.39}$$

Moving on to $n \geq 1$, one has

$$\hat{\delta}_\epsilon^{(-1)} S_n = \int d^D x \, \frac{\delta S_n}{\delta f^a{}_A} \bar{e}^a{}_B \epsilon^B{}_A + ST \tag{6.40}$$

$$\hat{\delta}_\epsilon^{(0)} S_{n-1} = \int d^D x \, \frac{\delta S_{n-1}}{\delta f^a{}_A} f^a{}_B \epsilon^B{}_A + ST \tag{6.41}$$

Substituting these in (6.37), one obtains

$$\int d^D x \left(\frac{\delta S_n}{\delta f^a{}_A} \bar{e}^{aB} + \frac{\delta S_{n-1}}{\delta f^a{}_A} f^{aB} \right) \epsilon_{BA} + ST = 0 \tag{6.42}$$

Following the same reasoning used to derive (6.39), we obtain

$$\frac{\delta S_n}{\delta f^a{}_{[A}} \bar{e}^{aB]} = -\frac{\delta S_{n-1}}{\delta f^a{}_{[A}} f^{aB]} = \frac{\delta S_{n-1}}{\delta \bar{e}^a{}_{[A}} \bar{e}^{aB]} \tag{6.43}$$

$$\Leftrightarrow \frac{\delta S_n}{\delta f^{[a}{}_A} \bar{e}_{b]A} = \frac{\delta S_{n-1}}{\delta \bar{e}^{[a}{}_A} \bar{e}_{b]A} \tag{6.44}$$

where took into account S_n invariance under LLTs by using (6.23) for the second equality.

Now we focus on the consequences of $S[e]$ being a (spacetime) scalar. Under (infinitesimal) GCTs in the active perspective, we have (up to ST)

$$\delta_\xi S[e] = 0 \quad \text{with} \quad \delta_\xi e^a{}_A = \xi^b \partial_b e^a{}_A - e^b{}_A \partial_b \xi^a = \xi^b \bar{\nabla}_b e^a{}_A - e^b{}_A \bar{\nabla}_b \xi^a \,.$$

Note that

$$\begin{aligned} \delta_\xi e^a{}_A \Big|_{e=\phi} &= \xi^b \bar{\nabla}_b \bar{e}^a{}_A - \bar{e}^b{}_A \bar{\nabla}_b \xi^a + \chi \xi^b \bar{\nabla}_b f^a{}_A - \chi f^b{}_A \bar{\nabla}_b \xi^a \\ &= \xi^b \bar{\omega}_{bAB} \bar{e}^{aB} - \bar{e}^b{}_A \bar{\nabla}_b \xi^a + \chi \xi^b \bar{\nabla}_b f^a{}_A - \chi f^b{}_A \bar{\nabla}_b \xi^a \end{aligned}$$

such that (again up to ST)

$$\delta_\xi S[\bar{e};f]=0 \quad \text{with} \quad \delta_\xi \bar{e}^a{}_A+\chi\delta_\xi f^a{}_A=\xi^b\bar{\omega}_{bAB}\bar{e}^{aB}-\bar{e}^b{}_A\bar{\nabla}_b\xi^a+\chi\xi^b\bar{\nabla}_b f^a{}_A-\chi f^b{}_A\bar{\nabla}_b\xi^a\ .$$

Since only $f^a{}_A$ is dynamical we choose $\delta_\xi\bar{e}^a{}_A=0$ and $\delta_\xi f^a{}_A=\chi^{-1}\delta_\xi^{(-1)}f^a{}_A+\delta_\xi^{(0)}f^a{}_A$ where

$$\begin{aligned}\delta_\xi^{(-1)}f^a{}_A&=\xi^b\bar{\omega}_{bAB}\bar{e}^{aB}-\bar{e}^b{}_A\bar{\nabla}_b\xi^a\\ \delta_\xi^{(0)}f^a{}_A&=\xi^b\bar{\nabla}_b f^a{}_A-f^b{}_A\bar{\nabla}_b\xi^a\end{aligned}\tag{6.45}$$

Similar to (6.35), we write $\delta_\xi S_n\equiv\chi^{-1}\delta_\xi^{(-1)}S_n+\delta_\xi^{(0)}S_n$ where $\delta_\xi^{(-1)}S_n=\frac{\delta S_n}{\delta f^a{}_A}\delta_\xi^{(-1)}f^a{}_A$ and $\delta_\xi^{(0)}S_n=\frac{\delta S_n}{\delta f^a{}_A}\delta_\xi^{(0)}f^a{}_A$. Hence, since S is invariant (up to ST) under GCT's:

$$\delta_\xi^{(-1)}S_0+ST=0\tag{6.46}$$

$$\delta_\xi^{(0)}S_{n-1}+\delta_\xi^{(0)}S_n+ST=0\ ,\quad n\geq 1\tag{6.47}$$

One has that

$$\delta_\xi^{(-1)}S_0=\int d^Dx\,\frac{\delta S_0}{\delta f^a{}_A}\delta_\xi^{(-1)}f^a{}_A+ST=\int d^Dx\left(\xi^b\bar{\omega}_{bAB}\bar{e}^{aB}\frac{\delta S_0}{\delta f^a{}_A}-\frac{\delta S_0}{\delta f^a{}_A}\bar{e}^b{}_A\bar{\nabla}_b\xi^a\right)+ST\tag{6.48}$$

Integrating the second term by parts and using (6.46), one obtains

$$\int d^Dx\left[\xi^a\bar{\omega}_{aAB}\bar{e}^{bB}\frac{\delta S_0}{\delta f^b{}_A}+\xi^a\bar{\nabla}_b\left(\frac{\delta S_0}{\delta f^a{}_A}\bar{e}^b{}_A\right)\right]+ST=0\tag{6.49}$$

Again, we convert the surface term into an integral over the boundary and, using an appropriate choice of boundary conditions for the gauge parameter ξ^a, it can be made to vanish. Due to arbitrariness of the parameter and (6.39), one arrives at the identity

$$\bar{\nabla}_b\left(\frac{\delta S_0}{\delta f^a{}_A}\bar{e}^b{}_A\right)=0\tag{6.50}$$

Moving on to (6.47), one has that

$$\delta_\xi^{(-1)}S_n=\int d^Dx\,\frac{\delta S_n}{\delta f^a{}_A}\delta_\xi^{(-1)}f^a{}_A+ST=\int d^Dx\left(\xi^b\bar{\omega}_{bAB}\bar{e}^{aB}\frac{\delta S_n}{\delta f^a{}_A}-\frac{\delta S_n}{\delta f^a{}_A}\bar{e}^b{}_A\bar{\nabla}_b\xi^a\right)+ST\ ,\tag{6.51}$$

where

$$\bar{\omega}_{bAB}\frac{\delta S_n}{\delta f^a{}_A}\bar{e}^{aB}=\bar{\omega}_{bAB}\frac{\delta S_n}{\delta f^a{}_{[A}}\bar{e}^{aB]}=-\bar{\omega}_{bAB}\frac{\delta S_{n-1}}{\delta f^a{}_{[A}}f^{aB]}\tag{6.52}$$

(using (6.43) on the second equality), and

$$\begin{aligned}\delta_\xi^{(0)}S_{n-1}&=\int d^Dx\,\frac{\delta S_{n-1}}{\delta f^a{}_A}\delta_\xi^{(0)}f^a{}_A+ST\\ &=\int d^Dx\left(\frac{\delta S_{n-1}}{\delta f^a{}_A}\xi^b\bar{\nabla}_b f^a{}_A-\frac{\delta S_{n-1}}{\delta f^a{}_A}f^b{}_A\bar{\nabla}_b\xi^a\right)+ST\end{aligned}\tag{6.53}$$

Using (6.47), integrating by parts to get rid of $\bar{\nabla}\epsilon$, one obtains

$$\int d^D x \left[\bar{\nabla}_b \left(\frac{\delta S_n}{\delta f^a{}_A}\bar{e}^b{}_A\right) - \bar{\omega}_{aAB}\frac{\delta S_{n-1}}{\delta f^b{}_A} f^{bB} + \frac{\delta S_{n-1}}{\delta f^b{}_A}\bar{\nabla}_a f^b{}_A + \bar{\nabla}_b \left(\frac{\delta S_{n-1}}{\delta f^a{}_A} f^b{}_A\right)\right]\xi^a + ST = 0 \tag{6.54}$$

Using (6.30) (without φ), we rewrite this as

$$\int d^D x \left[\bar{\nabla}_b \left(\frac{\delta S_n}{\delta f^a{}_A}\bar{e}^b{}_A\right)\xi^a - \bar{\nabla}_b \left(\frac{\delta S_{n-1}}{\delta \bar{e}^a{}_A}\bar{e}^b{}_A\right)\xi^a\right] + ST = 0 \tag{6.55}$$

Following the same reasoning used to derive (6.50), we obtain the identity

$$\bar{\nabla}^b \left(\frac{\delta S_n}{\delta f^a{}_A}\bar{e}_{bA} - \frac{\delta S_{n-1}}{\delta \bar{e}^a{}_A}\bar{e}_{bA}\right) = \bar{\nabla}^a \left(\frac{\delta S_n}{\delta f^a{}_A}\bar{e}_{bA} - \frac{\delta S_{n-1}}{\delta \bar{e}^a{}_A}\bar{e}_{bA}\right) = 0 \tag{6.56}$$

where we've used (6.44) in the first equality. Based on the extensive analysis we did of metric theories, it's pretty straightforward to extend some of our results to tetrad theories. Hereupon, the equation above is equivalent to

$$\begin{aligned}
&\frac{\delta S_n}{\delta f^a{}_A}\bar{e}_{bA} = \frac{\delta S_{n-1}}{\delta \bar{e}^a{}_A}\bar{e}_{bA} + \bar{\nabla}^c\bar{\nabla}^d\Psi_{[c(a][b)d]}[\bar{e}; f, \varphi] \\
&\Leftrightarrow \frac{\delta S_n}{\delta f^a{}_A} = \frac{\delta S_{n-1}}{\delta \bar{e}^a{}_A} + \bar{e}^{bA}\bar{\nabla}^c\bar{\nabla}^d\Psi_{[c(a][b)d]}[\bar{e}; f, \varphi] \equiv \left.\frac{\delta S_{n-1}}{\delta \gamma^a{}_A}\right|_{\gamma\to\bar{e}}
\end{aligned} \tag{6.57}$$

(6.56) also implies

$$\begin{aligned}
&\bar{\nabla}_b \sum_{n=1}^{\infty}\chi^n \frac{\delta S_n}{\delta f^a{}_A}\bar{e}^b{}_A = \bar{\nabla}_b \sum_{n=1}^{\infty}\chi^n \frac{\delta S_{n-1}}{\delta \bar{e}^a{}_A}\bar{e}^b{}_A \\
&\Leftrightarrow \bar{\nabla}_b \left(\frac{\delta S}{\delta f^a{}_A}\bar{e}^b{}_A\right) - \bar{\nabla}_b \left(\frac{\delta S_0}{\delta f^a{}_A}\bar{e}^b{}_A\right) = \chi\bar{\nabla}_b \left(\frac{\delta S}{\delta \bar{e}^a{}_A}\bar{e}^b{}_A\right) \\
&\Leftrightarrow \bar{\nabla}_b \left(\frac{\delta S}{\delta f^a{}_A}\bar{e}^b{}_A\right) = \bar{\nabla}_b \left(\frac{\delta S_0}{\delta f^a{}_A}\bar{e}^b{}_A\right) + \chi\bar{\nabla}_b T^{cb}_{\text{Ros}}\bar{g}_{ca}|\bar{e}|
\end{aligned} \tag{6.58}$$

6.3 Self-coupling of FP mass term

After being sure of the form the self-coupling condition takes in the case of a tetrad theory, we approach the main goal of this chapter: finding higher order self-interactions for the FP massive graviton. We start by writing (2.32) as

$$\mathscr{A}_h^{\mathrm{m}} = m^2 \int d^D x \left[\left(h^{ab}\eta_{ab}\right)^2 + k' h^{ab}\eta_{bc}h^{cd}\eta_{da}\right] = m^2 \int d^D x\, h^{ab}h^{ij}E_{abij}[\eta] \tag{6.59}$$

with $E_{abij}[\bar{g}] = \bar{g}_{ab}\bar{g}_{ij} + k'\bar{g}_{a(i}\bar{g}_{j)b}$, whose covariantization is

$$S_h^{\rm m} = m^2 \int d^D x \sqrt{-|\bar{g}|}\, h^{ab} h^{ij} E_{abij}[\bar{g}] \tag{6.60}$$

In this section, we fix $D = 4$ such that Lorentz indices go from 0 to 3 and we set $k' = -1$ since we're interested in the FP mass term. Let's perform the change of variables $\{\bar{g}^{ab}, h^{ab}\} \to \{\bar{e}^a{}_A, f^a{}_A\}$. Besides $\bar{g}^{ab} = \bar{e}^a{}_A \bar{e}^{bA}$, we choose

$$h^{ab} = f^a{}_A \bar{e}^{bA} \Leftrightarrow f^a{}_A = h^{ab}\bar{e}_{bA} \tag{6.61}$$

where f is constrained to obey

$$f^a{}_A \bar{e}^{bA} = f^b{}_A \bar{e}^{aA} \tag{6.62}$$

ensuring $h^{ab} = h^{ba}$ and this way we are exchanging $4\frac{4+1}{2} = 10$ independent variables in h by the same number in f. Recall that we go back to inertial coordinates $(\bar{g}^{ab} \to \eta^{ab})$ by bringing $\bar{e}^a{}_A$ to $\delta^a_A \equiv \bar{\delta}^a{}_A$. One sees that S_h, the covariantization of the Diff invariant action $\mathscr{A}_h$, after the change of variables is gauge invariant under

$$\delta f^a{}_A = \bar{e}_{bA}\delta h^{ab} = 2\bar{e}_{bA}\bar{\nabla}^{(b}\xi^{a)} \tag{6.63}$$

and one can derive the corresponding Gauge/Bianchi Identity. Note that, since $h^{cd} = h^{dc}$,

$$\frac{\delta S_h}{\delta f^a{}_A}\bar{e}_{bA} = \bar{e}_{bA}\int d^D y \frac{\delta S_h}{\delta h^{cd}(y)}\frac{\delta h^{cd}(y)}{\delta f^a{}_A} = \bar{e}_{bA}\int d^D y \frac{\delta S_h}{\delta h^{cd}}\delta^c_a e^{dA}\delta(x-y) = \frac{\delta S_h}{\delta h^{cd}}\delta^c_a \delta^d_b \tag{6.64}$$

is symmetric under $a \leftrightarrow b$ (c.f. (6.39)). Hence, we have

$$\delta S_h = 2\int d^D x \frac{\delta S_h}{\delta f^a{}_A}\bar{e}_{bA}\bar{\nabla}^b \xi^a + ST = 0 \tag{6.65}$$

such that, integrating by parts, making surface terms vanish as usual and using the arbitrariness of the parameter ξ, we obtain the off-shell identity

$$\bar{\nabla}^b \left(\frac{\delta S_h}{\delta f^a{}_A}\bar{e}_{bA}\right) = 0 \tag{6.66}$$

like we expected from (6.50). Moving on to the mass term, that is the centre of this section, we have

$$\begin{aligned}\sqrt{-|\bar{g}|}h^{ab}h^{ij}E_{abij}[\bar{g}] &= |\bar{e}| f^a{}_A f^i{}_C (\bar{e}_a{}^A \bar{e}_i{}^C - \frac{1}{2}\bar{g}_{ai}\eta^{AC} - \frac{1}{2}\bar{e}_i{}^A \bar{e}_a{}^C) \\ &= |\bar{e}|\big(f^a{}_A \bar{e}_a{}^A f^i{}_C \bar{e}_i{}^C - f^a{}_A \bar{e}_a{}^C f^i{}_C \bar{e}_i{}^A\big) = 2|\bar{e}| f_a{}^A f_i{}^C \bar{e}^{[a}{}_A \bar{e}^{i]}{}_C \equiv \mathscr{L}_{\rm FP}[\bar{e}; f]\end{aligned} \tag{6.67}$$

where we used contraint (6.62)[24] for the second equality ($f^a{}_A \bar{e}_i{}^A \bar{e}_a{}^C = f^a{}_A \bar{g}_{ai}\eta^{AC}$). In terms of our new variables, $\mathscr{A}_h^{\mathrm{m}}$ is

$$\mathscr{A}_{\mathrm{FP}} \equiv m^2 \int d^D x\, \mathscr{L}_{\mathrm{FP}}[\bar{\delta}; f] = m^2 \int d^D x\, f_a{}^A f_b{}^B F^{ab}{}_{AB}[\bar{\delta}] \tag{6.68}$$

where $F^{ab}{}_{AB}[\bar{\delta}] \equiv 2\bar{\delta}^{[a}{}_A \bar{\delta}^{b]}{}_B |\bar{\delta}|$. If we covariantize $\mathscr{A}_{\mathrm{FP}}$, according to the definition we estabilished in section 2.1 and have been employing since then, we obtain

$$S_0^{\mathrm{FP}}[f;\bar{e}] = m^2 \int d^D x f_a{}^A f_b{}^B F^{ab}{}_{AB}[\bar{e}] \tag{6.69}$$

which coincides with S_h^{m}. However, with our current variables and specifically with the FP mass term, we also have

$$\hat{S}_0^{\mathrm{FP}}[f;\bar{e}] = m^2 \int d^D x\, f_a{}^A f_b{}^B \hat{F}^{ab}{}_{AB}[\bar{e}] \tag{6.70}$$

where $2\hat{F}^{ab}{}_{AB}[\bar{e}] = \bar{e}_c{}^C \bar{e}_d{}^D \epsilon_{ABCD}\tilde{\epsilon}^{abcd}$ (note that $\epsilon_{ABCD} = \tilde{\epsilon}_{ABCD}$[25]), since

$$\hat{F}^{ab}{}_{AB}[\bar{\delta}] = \frac{1}{2}\epsilon_{ABCD}\tilde{\epsilon}^{abCD} = \frac{-1}{2}\epsilon_{ABCD}\epsilon^{abCD}\sqrt{-|\eta|} = \frac{-1}{2}\epsilon_{ABCD}\epsilon^{abCD}|\bar{\delta}| = 2|\bar{\delta}|\delta^{[a}_A\delta^{b]}_B \tag{6.71}$$

Let us now focus on the self-coupling condition (6.57) using both covariantizations (since the mass terms are zero order in derivatives we set $\Psi = 0$). Starting with $\hat{S}_0^{\mathrm{FP}}$:

$$\frac{\delta \hat{S}_0^{\mathrm{FP}}}{\delta \bar{e}_q{}^Q} = m^2 f_a{}^A f_b{}^B \bar{e}_c{}^C \epsilon_{ABCQ}\tilde{\epsilon}^{abcq} \equiv m^2 f_a{}^A f_b{}^B \hat{N}^{abq}{}_{ABQ} \tag{6.72}$$

$$\Rightarrow \frac{\delta \hat{S}_0^{\mathrm{FP}}}{\delta \bar{e}_q{}^Q}\bigg|_{\bar{e}=\bar{\delta}} = m^2 f_a{}^A f_b{}^B \epsilon_{ABQC}\tilde{\epsilon}^{abqC} = -m^2|\bar{\delta}| f_a{}^A f_b{}^B \epsilon_{ABQC}\epsilon^{abqC} = 6m^2|\bar{\delta}| f_a{}^A f_b{}^B \delta^{[a}_A\delta^b_B\delta^{q]}_Q \tag{6.73}$$

Inertial coordinates will be useful as a means of comparison between both covariantized actions. Moving on to S_0^{FP}:

$$\begin{aligned}
\frac{\delta S_0^{\mathrm{FP}}}{\delta \bar{e}_q{}^Q} &= m^2 f_a{}^A f_b{}^B \left[\frac{\partial |\bar{e}|}{\partial \bar{e}_q{}^Q}\left(\bar{e}^a{}_A\bar{e}^b{}_B - \bar{e}^b{}_A\bar{e}^a{}_B\right) + |\bar{e}|\frac{\partial\left(\bar{e}^a{}_A\bar{e}^b{}_B - \bar{e}^b{}_A\bar{e}^a{}_B\right)}{\partial \bar{e}^t{}_T}\eta_{QT}\bar{g}^{qt}\right] \\
&= m^2 f_a{}^A f_b{}^B |\bar{e}|\left(\bar{e}^a{}_A\bar{e}^b{}_B\bar{e}^q{}_Q - \bar{e}^b{}_A\bar{e}^a{}_B\bar{e}^q{}_Q + \bar{e}^a{}_A\bar{g}^{bq}\eta_{BQ} + {}^a_A \leftrightarrow {}^b_B - \bar{e}^b{}_A\bar{g}^{aq}\eta_{BQ} - {}^a_A \leftrightarrow {}^b_B\right) \\
&= m^2 f_a{}^A f_b{}^B |\bar{e}|\left(\bar{e}^a{}_A\bar{e}^b{}_B\bar{e}^q{}_Q - \bar{e}^b{}_A\bar{e}^a{}_B\bar{e}^q{}_Q + \bar{e}^a{}_A\bar{e}^b{}_Q\bar{e}^q{}_B + {}^a_A \leftrightarrow {}^b_B - \bar{e}^a{}_B\bar{e}^q{}_A\bar{e}^b{}_Q - {}^a_A \leftrightarrow {}^b_B\right) \\
&= 2m^2 f_a{}^A f_b{}^B |\bar{e}|\left(\bar{e}^{[a}{}_A\bar{e}^{b]}{}_B\bar{e}^q{}_Q + \bar{e}^{[a|}{}_A\bar{e}^q{}_B\bar{e}^{|b]}{}_Q - \bar{e}^q{}_A\bar{e}^{[a}{}_B\bar{e}^{b]}{}_Q\right) \equiv m^2 f_a{}^A f_b{}^B N'^{abq}{}_{ABQ}
\end{aligned} \tag{6.74}$$

[24]When possibel we use (6.62) to get rid of contractions between f's.

[25]Our conventions are such that: $\epsilon_{a_1\ldots a_n} = \sqrt{-|\bar{g}|}\tilde{\epsilon}_{a_1\ldots a_n}$; $\epsilon^{a_1\ldots a_n} = \epsilon_{a_1\ldots a_n} g^{a_1b_1}\ldots g^{a_nb_n}$; $\epsilon^{a_1\ldots a_n} = \frac{sgn(|\bar{g}|)}{\sqrt{-|\bar{g}|}}\tilde{\epsilon}^{a_1\ldots a_n}$ where $\tilde{\epsilon}^{a_1\ldots a_n}$ is numerically identical to $\tilde{\epsilon}_{a_1\ldots a_n}$; and $\epsilon_{a_1\ldots a_p\ldots b_1\ldots b_{n-p}}\epsilon^{a_1\ldots a_p\ldots c_1\ldots c_{n-p}} = sgn(|\bar{g}|)p!\,(n-p)!\,\delta^{[c_1}_{b_1}\ldots\delta^{c_{n-p}]}_{b_{n-p}}$ (anti-symmetrization here concerns all indices between the ones next to the parentheses).

Constraint (6.62) was used above to get rid of $\bar{g}$ and η. Taking $\bar{e} = \bar{\delta}$ we see that this is different from (6.73). Similar to what we did in sections 4.2 and 5.2, we see that

$$\frac{\delta S_1^{\mathrm{FP}}}{\delta f_q{}^Q} = \frac{\delta S_0^{\mathrm{FP}}}{\delta \bar{e}_q{}^Q} \tag{6.75}$$

imposes that S_1^{FP} has the following form[26]:

$$S_1^{\mathrm{FP}} = \frac{m^2}{3} \int d^D x\, f_a{}^A f_b{}^B f_c{}^C N^{abc}{}_{ABC} \tag{6.76}$$

This makes $N^{abc}{}_{ABC}$ symmetric under any permutation of $\{{}^a_A, {}^b_B, {}^c_C\}$. With or without assuming it, this would apply to N in the derivative below, or any other object in which N enters through expression (6.76). We thus have

$$\frac{\delta S_1^{\mathrm{FP}}}{\delta f_q{}^Q} = m^2 f_a{}^A f_b{}^B N^{abq}{}_{ABQ} \tag{6.77}$$

However $N'^{abq}{}_{ABQ}$ is not symmetric under permutations of $\{{}^a_A, {}^b_B, {}^q_Q\}$, making $\mathscr{A}^0_{\mathrm{FP}}$ unable to satisfy the self-coupling condition (6.75). This time, non-minimal couplings can't come to the rescue since the mass term is free from derivatives. It's great that we have also found covariantization $\hat{S}_0^{\mathrm{FP}}$, preventing us from abandoning the possibility of self-coupling with the FP mass term. From

$$\frac{\delta \hat{S}_1^{\mathrm{FP}}}{\delta f_q{}^Q} = \frac{\delta \hat{S}_0^{\mathrm{FP}}}{\delta \bar{e}_q{}^Q} \tag{6.78}$$

it's easy to see that

$$\hat{S}_1^{\mathrm{FP}} = \frac{m^2}{3} \int d^D x\, f_a{}^A f_b{}^B f_c{}^C \hat{N}^{abc}{}_{ABC} \tag{6.79}$$

Note that $\hat{N}^{abq}{}_{ABQ} = \bar{e}_c{}^C \epsilon_{ABCQ} \tilde{\epsilon}^{abcq}$ is symmetric under permutations of $\{{}^a_A, {}^b_B, {}^c_C\}$. You might have noticed that we haven't used (6.57) accurately, which should be

$$\frac{\delta S_{n+1}^{\mathrm{FP}}}{\delta f^a{}_A} = \frac{\delta S_n^{\mathrm{FP}}}{\delta \bar{e}^a{}_A} \Leftrightarrow \frac{\delta S_{n+1}^{\mathrm{FP}}}{\delta f_d{}^D} \eta^{AD} \bar{g}_{ad} = -\frac{\delta S_n^{\mathrm{FP}}}{\delta \bar{e}_b{}^B} \bar{e}_b{}^A \bar{e}_a{}^B \Leftrightarrow \frac{\delta S_{n+1}^{\mathrm{FP}}}{\delta f_a{}^A} = -\frac{\delta S_n^{\mathrm{FP}}}{\delta \bar{e}_b{}^B} \bar{e}_{bA} \bar{e}^{aB} \tag{6.80}$$

such that instead of (6.75) we should have

$$\frac{\delta S_1^{\mathrm{FP}}}{\delta f_q{}^Q} = -\frac{\delta S_0^{\mathrm{FP}}}{\delta \bar{e}_c{}^C} \bar{e}_{cQ} \bar{e}^{qC} = -m^2 f_a{}^A f_b{}^B N'^{abc}{}_{ABC} \bar{e}_{cQ} \bar{e}^{qC} \tag{6.81}$$

[26]In this section, an unspecified capital latin letter with indices may include not only the tetrad and its inverse, but also the epsilon symbol.

where (6.74) was used. Note that (6.76) can also be written as

$$\frac{m^2}{3}\int d^Dx\, f_a{}^A f_b{}^B f_q{}^Q N^{abc}{}_{ABC}\bar{e}_{cQ}\bar{e}^{qC} \tag{6.82}$$

where constraint (6.62) was used. Differentiating S_1^{FP}, as given by (6.76) and (6.82), w.r.t. $f_q{}^Q$ we obtain

$$\frac{\delta S_1^{\mathrm{FP}}}{\delta f_q{}^Q} = -\frac{m^2}{3} f_a{}^A f_b{}^B N^{abq}{}_{ABQ} - \frac{m^2}{3} f_a{}^A f_c{}^C N^{aqc}{}_{AQC} - \frac{m^2}{3} f_b{}^B f_c{}^C N^{qbc}{}_{QBC} \tag{6.83}$$

and

$$\begin{aligned}\frac{\delta S_1^{\mathrm{FP}}}{\delta f_q{}^Q} &= -\frac{m^2}{3} f_a{}^A f_b{}^B N^{abc}{}_{ABC}\bar{e}_{cQ}\bar{e}^{qC} - \frac{m^2}{3} f_a{}^A f_t{}^T N^{aqc}{}_{AQC}\bar{e}_{cT}\bar{e}^{tC} - \frac{m^2}{3} f_b{}^B f_t{}^T N^{qbc}{}_{QBC}\bar{e}_{cT}\bar{e}^{tC}\\ &= -\frac{m^2}{3} f_a{}^A f_b{}^B N^{abc}{}_{ABC}\bar{e}_{cQ}\bar{e}^{qC} - \frac{m^2}{3} f_a{}^A f_c{}^C N^{aqc}{}_{AQC} - \frac{m^2}{3} f_b{}^B f_c{}^C N^{qbc}{}_{QBC}\end{aligned} \tag{6.84}$$

respectively. Comparing these we reach the identity

$$f_a{}^A f_b{}^B N^{abc}{}_{ABC}\bar{e}_{cQ}\bar{e}^{qC} = f_a{}^A f_b{}^B N^{abq}{}_{ABQ} \tag{6.85}$$

such that we can rewrite the self-coupling condition (6.81):

$$\frac{\delta S_1^{\mathrm{FP}}}{\delta f_q{}^Q} = -m^2 f_a{}^A f_b{}^B N'^{abq}{}_{ABQ} \tag{6.86}$$

Hence, like before, it's not possible to solve the self-coupling condition with S_0^{FP} since $N'^{abq}{}_{ABQ}$ lacks the required symmetry. We have to call upon $\hat{S}_0^{\mathrm{FP}}$ covariantization, since $\hat{N}^{abq}{}_{ABQ}$ possesses that symmetry:

$$\begin{aligned}\frac{\delta \hat{S}_1^{\mathrm{FP}}}{\delta f_q{}^Q} = -\frac{\delta \hat{S}_0^{\mathrm{FP}}}{\delta \bar{e}_c{}^C}\bar{e}_{cQ}\bar{e}^{qC} &= -m^2 f_a{}^A f_b{}^B \hat{N}^{abc}{}_{ABC}\bar{e}_{cQ}\bar{e}^{qC}\\ &= -m^2 f_a{}^A f_b{}^B \hat{N}^{abq}{}_{ABQ} = -\frac{\delta \hat{S}_0^{\mathrm{FP}}}{\delta \bar{e}_q{}^Q}\end{aligned} \tag{6.87}$$

Therefore, we obtain

$$\hat{S}_1^{\mathrm{FP}} = -\frac{m^2}{3}\int d^Dx\, f_a{}^A f_b{}^B f_c{}^C \hat{N}^{abc}{}_{ABC} = -\frac{m^2}{3}\int d^Dx\, f_a{}^A f_b{}^B f_c{}^C \bar{e}_d{}^D \epsilon_{ABCD}\tilde{\epsilon}^{abcd} \tag{6.88}$$

which is minus our previous (wrong) expression for $\hat{S}_1^{\rm FP}$. The same happens with the next step of the iterative procedure:

$$
\begin{aligned}
&\frac{\delta \hat{S}_1^{\rm FP}}{\delta \bar{e}_q{}^Q} = -\frac{m^2}{3} f_a{}^A f_b{}^B f_c{}^C \epsilon_{ABCQ}\tilde{\epsilon}^{abcq} \\
&\frac{\delta \hat{S}_2^{\rm FP}}{\delta f_q{}^Q} = -\frac{\delta \hat{S}_1^{\rm FP}}{\delta \bar{e}_c{}^C}\bar{e}_{cQ}\bar{e}^{qC} \Rightarrow S_2^{\rm FP} = \frac{m^2}{12}\int d^Dx\, f_a{}^A f_b{}^B f_c{}^C f_d{}^D \epsilon_{ABCD}\tilde{\epsilon}^{abcd} \Rightarrow \frac{\delta \hat{S}_2^{\rm FP}}{\delta f_q{}^Q} = -\frac{\delta \hat{S}_1^{\rm FP}}{\delta \bar{e}_q{}^Q} \\
&\frac{\delta \hat{S}_2^{\rm FP}}{\delta \bar{e}_q{}^Q} = 0
\end{aligned}
\tag{6.89}
$$

The last equation above makes the self-coulpling conditions in further steps of the iterative procedure trivial. Hence we've arrived at

$$
\sum_{n=0}^{\infty} \chi^n \hat{S}_n^{\rm FP} = \hat{S}_0^{\rm FP} + \chi \hat{S}_1^{\rm FP} + \chi^2 \hat{S}_2^{\rm FP} \equiv \hat{S}^{\rm FP} \tag{6.90}
$$

We have that

$$
\frac{\delta \hat{S}^{\rm FP}}{\delta \bar{e}^q{}_Q} = \frac{\delta \hat{S}_0^{\rm FP}}{\delta \bar{e}^q{}_Q} + \chi \frac{\delta \hat{S}_1^{\rm FP}}{\delta \bar{e}^q{}_Q} = \frac{\delta \hat{S}_1^{\rm FP}}{\delta f^q{}_Q} + \chi \frac{\delta \hat{S}_2^{\rm FP}}{\delta f^q{}_Q} \tag{6.91}
$$

and

$$
\frac{\delta \hat{S}^{\rm FP}}{\delta f^q{}_Q} = \frac{\delta \hat{S}_0^{\rm FP}}{\delta f^q{}_Q} + \chi \frac{\delta \hat{S}_1^{\rm FP}}{\delta f^q{}_Q} + \chi^2 \frac{\delta \hat{S}_2^{\rm FP}}{\delta f^q{}_Q} = \frac{\delta \hat{S}_0^{\rm FP}}{\delta f^q{}_Q} + \chi \frac{\delta \hat{S}^{\rm FP}}{\delta \bar{e}^q{}_Q} \tag{6.92}
$$

which implies (6.58), such that the equation of motion for the total action leads to the field at a linear level (graviton) being coupled to the stress-energy tensor of the theory associated to the total action. What we have ended up with is equivalent (as shown in [42]) to dRGT theory of ghost-free massive gravity [4, 5] with less free parameters:

$$
\begin{aligned}
\hat{S}^{\rm FP} = m^2 \int d^Dx \Bigg(&\frac{1}{2} f_a{}^A f_b{}^B \bar{e}_c{}^C \bar{e}_d{}^D \epsilon_{ABCD}\tilde{\epsilon}^{abcd} - \frac{\chi}{3} f_a{}^A f_b{}^B f_c{}^C \bar{e}_d{}^D \epsilon_{ABCD}\tilde{\epsilon}^{abcd} \\
&+ \frac{\chi^2}{12} f_a{}^A f_b{}^B f_c{}^C f_d{}^D \epsilon_{ABCD}\tilde{\epsilon}^{abcd} \Bigg)
\end{aligned}
\tag{6.93}
$$

7 Conclusion

We would like to conclude with a few important remarks that deserve some elaboration. The Bi-connection formulation of GR (and its unimodular variation) is particularly appealing. Seeing background independence as the "gauge symmetry" of Einstein's theory, it's reasonable that when we probe it by making an experiment there is a specific gauge that we can choose to make easier fitting the results into our theoretical framework. Due to the local nature of our physical experiments, this choice of gauge consists in selecting a background and linearising around it. For a theoretical physicist, the information generated by these experiments (for spin-2 this is the detection of gravitational waves) is stored in the SRFTs that have been developed throughout the last century and Bi-connection GR allows us to handle background independence explicitly through the auxiliary connection $\check{\Gamma}$! However before anyone thinks that our "experiments" in flat spacetime are uniquely consistent with GR/background independence, let us rewrite (4.29) and (4.33), respectively, in such a manner that

$$\left.\frac{\delta\tilde{S}_0}{\delta\gamma^{pq}}\right|_{\gamma\to\bar{g}} = -\sqrt{-|\bar{g}|}\left(\frac{1}{2}A_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq} + B_{pq}{}^{c}{}_{ef}{}^{d}{}_{ab}\right)[\bar{g}]\bar{\nabla}_c h^{ab}\bar{\nabla}_d h^{ef} - \sqrt{-|\bar{g}|}B_{pq}{}^{c}{}_{ef}{}^{d}{}_{ab}[\bar{g}]h^{ab}\bar{\nabla}_c\bar{\nabla}_d h^{ef}$$

and

$$\frac{\delta S_1}{\delta h^{pq}} = \sqrt{-|\bar{g}|}\left(X_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq} - 2X_{pq}{}^{c}{}_{ef}{}^{d}{}_{ab}\right)[\bar{g}]\bar{\nabla}_c h^{ab}\bar{\nabla}_d h^{ef} - 2\sqrt{-|\bar{g}|}X_{pq}{}^{c}{}_{ef}{}^{d}{}_{ab}[\bar{g}]h^{ab}\bar{\nabla}_c\bar{\nabla}_d h^{ef}$$

where $S_1 = \int d^Dx\sqrt{-|\bar{g}|}X_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq}[\bar{g}]h^{pq}\bar{\nabla}_c h^{ab}\bar{\nabla}_d h^{ef}$. This way, since $\bar{\nabla}_{[c}\bar{\nabla}_{d]} = 0$, it becomes obvious that comparing the equations above before bringing them to the flat limit, like we did in (4.34), is not one hundred percent equivalent to the self-coupling condition (and the same applies beyond the first step of the iterative procedure). Instead, we should have had

$$\frac{\delta S_1}{\delta h^{pq}} = \left.\frac{\delta\tilde{S}_0}{\delta\gamma^{pq}}\right|_{\gamma\to\bar{g}} \Rightarrow \begin{cases} X_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq} - 2X_{pq}{}^{c}{}_{ef}{}^{d}{}_{ab} = -\frac{1}{2}A_{ab}{}^{c}{}_{ef}{}^{d}{}_{pq} - B_{pq}{}^{c}{}_{ef}{}^{d}{}_{ab} \\ 2X_{pq}{}^{(c}{}_{ef}{}^{d)}{}_{ab} = B_{pq}{}^{(c}{}_{ef}{}^{d)}{}_{ab} \end{cases}$$

$$\Leftrightarrow \begin{cases} X_{ab}{}^{(c}{}_{ef}{}^{d)}{}_{pq} = -\frac{1}{2}A_{ab}{}^{(c}{}_{ef}{}^{d)}{}_{pq} \\ B_{pq}{}^{(c}{}_{ef}{}^{d)}{}_{ab} + A_{pq}{}^{(c}{}_{ef}{}^{d)}{}_{ab} = 0 \\ X_{ab}{}^{[c}{}_{ef}{}^{d]}{}_{pq} - 2X_{pq}{}^{[c}{}_{ef}{}^{d]}{}_{ab} = -\frac{1}{2}A_{ab}{}^{[c}{}_{ef}{}^{d]}{}_{pq} - B_{pq}{}^{[c}{}_{ef}{}^{d]}{}_{ab} \end{cases}$$

such that before what we've actually done was solving the last equation by demanding that the first and second terms on each side equate independently. There should be a priori other solutions and we can thus infer that the self-coupling programme of FP SRFT does not lead uniquely to GR. However Einstein's theory is set apart due to background independence.

The reason why we previously concealed this "small" mistake is that it allowed us to conduct the programme in a way that suggest the following conclusion: to non-geometrically DERIVE Einstein's theory of General Relativiy, we would have to compare our SRFTs with the local theories by physicists from another planet. But not any planet: they must inhabit a region of the universe where curvature can't be neglected! (So that they would self-couple a quadratic action with non-commuting derivatives).

Let us end by mentioning some ways this work could be extended, like its relation to the equivalence principles of GR and to the existence of a local concept of gravitational energy. For completeness, it would be interesting to include the SRFT invariant under WTDiff transformations [3] and consequently augmenting the sorts of "covariantizations" available. Lastly, the author would like in future work to consider spinor matter fields and also inspect the impact of field redefinitions on section 2.3 and on the iterative programme in general.

A Additional ambiguity in special-relativistic action

Diff gauge invariance forces $a = 1 = b$ in (2.17), as seen back at section 2.2, but there is still some ambiguity (which we are going to parametrise through $\mathfrak{s} \in \mathbb{R}$) in the way we can write this action:

$$\mathscr{A} = \int d^D x \left[\frac{-1}{4} \partial_a h_{bc} \partial^a h^{bc} + \frac{1}{2} \Big(\mathfrak{s} \partial^b h_{bc} \partial_a h^{ac} + (1-\mathfrak{s}) \partial_a h_{bc} \partial^b h^{ac} \Big) - \frac{1}{2} \partial_a h^{ab} \partial_b h + \frac{1}{4} \partial_a h \partial^a h \right] \tag{A.1}$$

since

$$\int d^D x \, \partial_a h_{bc} \partial^b h^{ac} = \int d^D x \, \partial^b h_{bc} \partial_a h^{ac} + ST \tag{A.2}$$

Covariantising this action and minimal coupling it, we get

$$\begin{aligned} \tilde{S}_0 &= \frac{-1}{2} \int d^D x \sqrt{-|\gamma|} \nabla_c h^{ab} \nabla_d h^{ef} \left(K_{ab}{}^{c}{}_{ef}{}^{d} + \mathfrak{s} P_a{}^{[cd]}{}_{bef} \right) \\ &\equiv \frac{-1}{2} \int d^D x \sqrt{-|\gamma|} \nabla_c h^{ab} \nabla_d h^{ef} K^{(\mathfrak{s})}{}_{ab}{}^{c}{}_{ef}{}^{d} \end{aligned} \tag{A.3}$$

where $P_a{}^{cd}{}_{bef} \equiv \delta^c_e \gamma_{fa} \delta^d_b$. The question that immediatly arises is: if we then start the iterative procedure, will we end up with an action similar to (4.54) with $K^{(\mathfrak{s})}$ instead of K? This would differ from (4.54) by more than surface terms and violate background independence.

The self-coupling condtion in the first step of the procedure leads again to (4.34) where A and B, appart from depending on $K^{(\mathfrak{s})}$ instead of K, are given by the same expressions as in section 4.2. Can we satisfy the consistency requirement with an arbitrary $\mathfrak{s}$ (we already know this is possible when $\mathfrak{s}$ vanishes)? We'll now see that we can in the first step but in the following ones there seems to be no solution for the self-coupling condition if $\mathfrak{s} \neq 0$: using $\nabla_{[c} \nabla_{d]} h^{ab} = 2 \mathcal{R}^{(a}{}_{icd} h^{b)i} \Leftarrow \nabla_{[c} \nabla_{d]} v^a = \mathcal{R}^a{}_{icd} v^i$ [27], we get

$$\begin{aligned} &\int d^D x \sqrt{-|\gamma|} \, P_a{}^{[cd]}{}_{bef} \nabla_d h^{ef} \nabla_c h^{ab} = \int d^D x \sqrt{-|\gamma|} \, P_a{}^{cd}{}_{bef} h^{ef} \nabla_{[c} \nabla_{d]} h^{ab} + ST \\ &= 2 \int d^D x \sqrt{-|\gamma|} \, P_a{}^{cd}{}_{bef} \mathcal{R}^{(a}{}_{icd} h^{b)i} h^{ef} + ST \end{aligned} \tag{A.4}$$

which substitutes (A.2) (note that $\mathcal{R}$ vanishes when γ goes to $\bar{g}$). This can be written as

$$S_0^{\text{NM}} = \int d^D x \sqrt{-|\gamma|} \, Q_a{}^{icd}{}_{bjef} \mathcal{R}^a{}_{icd} h^{bj} h^{ef} \tag{A.5}$$

[27] The last equality is equivalent to (4.79).

if $Q_a{}^{icd}{}_{bjef} = 2\delta^i_j P_{(a}{}^{cd}{}_{b)ef}$. Making use of a non-minimal coupling term proportional to this, one expects to start the second step of the iterative procedure with

$$\begin{aligned}\tilde{S}_1 &= \frac{-1}{2}\int d^D x\, \frac{\partial\sqrt{-|\gamma|}K^{(\mathfrak{s})}{}_{ab}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{st}} h^{st}\nabla_c h^{ab}\nabla_d h^{ef} \\ &\equiv \frac{-1}{2}\int d^D x\, \frac{\partial\sqrt{-|\gamma|}K_{ab}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{st}} h^{st}\nabla_c h^{ab}\nabla_d h^{ef} + \mathfrak{s}\int d^D x\, P_a{}^{[cd]}{}_{befst} h^{st}\nabla_c h^{ab}\nabla_d h^{ef}\end{aligned} \tag{A.6}$$

The present consistency requirement could be similarly satisfied using

$$\begin{aligned}S_1^{\rm NM} &= \int d^D x\sqrt{-|\gamma|}Q_a{}^{icd}{}_{bjefst}\mathcal{R}^a{}_{icd}h^{bj}h^{ef}h^{st} \equiv 2\int d^D x P_{(a}{}^{cd}{}_{b)efst}\mathcal{R}^a{}_{icd}h^{bi}h^{ef}h^{st} \\ &= \int d^D x\, P_a{}^{cd}{}_{befst}h^{st}h^{ef}\nabla_{[c}\nabla_{d]}h^{ab} = 2\int d^D x\, P_a{}^{[cd]}{}_{befst}h^{st}\nabla_d h^{ef}\nabla_c h^{ab} + ST\end{aligned} \tag{A.7}$$

were it not for the fact that this requires $P_a{}^{[cd]}{}_{befst} = P_a{}^{[cd]}{}_{bstef}$ which is false.

B Connecting Canonical and Rosenfeld's EMT (alternative)

One can write the r.h.s. of (3.24), taking into account that $\frac{\partial\mathscr{L}}{\partial\bar{\nabla}_c h^{ab}}\delta h^{ab} = \frac{\partial\mathscr{L}}{\partial\partial_c h^{ab}}\delta h^{ab}$ and $\frac{\partial\mathscr{L}}{\partial\bar{\nabla}_c\varphi}\delta\varphi = \frac{\partial\mathscr{L}}{\partial\partial_c\varphi}\delta\varphi$ are weight-1 vector densities, as

$$\delta\mathscr{L} = \frac{\delta S}{\delta h^{ab}}\delta h^{ab} + \frac{\delta S}{\delta\varphi}\delta\varphi + \frac{\delta S}{\delta\bar{g}^{ab}}\delta\bar{g}^{ab} + \bar{\nabla}_c\left(\frac{\partial\mathscr{L}}{\partial\bar{\nabla}_c h^{ab}}\delta h^{ab} + \frac{\partial\mathscr{L}}{\partial\bar{\nabla}_c\varphi}\delta\varphi\right) + \partial_c\left(\frac{\partial\mathscr{L}}{\partial\partial_c\bar{g}^{ab}}\delta\bar{g}^{ab}\right)\ . \tag{B.1}$$

Actually, $\frac{\partial\mathscr{L}}{\partial\bar{\nabla}_c\bar{g}^{ab}}\delta\bar{g}^{ab}$ is a weight-1 vector density and so the coordinate derivative in front of the last term above can also be replaced by the covariant one (see [34]). Hence,

$$\delta\mathscr{L} = \frac{\delta S}{\delta h^{ab}}\delta h^{ab} + \frac{\delta S}{\delta\varphi}\delta\varphi + \frac{\delta S}{\delta\bar{g}^{ab}}\delta\bar{g}^{ab} + \bar{\nabla}_c\left(\frac{\partial\mathscr{L}}{\partial\bar{\nabla}_c h^{ab}}\delta h^{ab} + \frac{\partial\mathscr{L}}{\partial\bar{\nabla}_c\varphi}\delta\varphi + \frac{\partial\mathscr{L}}{\partial\partial_c\bar{g}^{ab}}\delta\bar{g}^{ab}\right) \tag{B.2}$$

Now, we are also going to specify the variations of the dynamical fields. Starting with $\delta\bar{g}^{ab}$ and

$$\begin{aligned}\delta h^{ab} &= \xi^c\partial_c h^{ab} - 2h^{c(a}\partial_c\xi^{b)} \\ &= \xi^c\bar{\nabla}_c h^{ab} - 2h^{c(a}\bar{\nabla}_c\xi^{b)}\ ,\end{aligned} \tag{B.3}$$

we can already write most of the terms in (B.2) in terms of the transformation parameter:

$$\frac{\delta S}{\delta h^{ab}}\delta h^{ab} =\frac{\delta S}{\delta h^{ab}}\xi^c\bar{\nabla}_c h^{ab} - 2\frac{\delta S}{\delta h^{ab}}h^{ca}\bar{\nabla}_c\xi^b \tag{B.4}$$

$$\begin{aligned}\bar{\nabla}_i\left(\frac{\partial\mathscr{L}}{\partial\bar{\nabla}_i h^{ab}}\delta h^{ab}\right) =&\bar{\nabla}_i\left(\frac{\partial\mathscr{L}}{\partial\bar{\nabla}_i h^{ab}}\xi^c\bar{\nabla}_c h^{ab}\right) - 2\bar{\nabla}_i\left(\frac{\partial\mathscr{L}}{\partial\bar{\nabla}_i h^{ab}}h^{ca}\bar{\nabla}_c\xi^b\right)\\ =&\bar{\nabla}_i\left(\frac{\partial\mathscr{L}}{\partial\bar{\nabla}_i h^{ab}}\bar{\nabla}_c h^{ab}\right)\xi^c + \left(\frac{\partial\mathscr{L}}{\partial\bar{\nabla}_i h^{ab}}\bar{\nabla}_c h^{ab}\right)\bar{\nabla}_i\xi^c\\ &- 2\bar{\nabla}_i\left(\frac{\partial\mathscr{L}}{\partial\bar{\nabla}_i h^{ab}}h^{ca}\right)\bar{\nabla}_c\xi^b - 2\left(\frac{\partial\mathscr{L}}{\partial\bar{\nabla}_i h^{ab}}h^{ca}\right)\bar{\nabla}_i\bar{\nabla}_c\xi^b\end{aligned} \tag{B.5}$$

$$\frac{\delta S}{\delta\bar{g}^{ab}}\delta\bar{g}^{ab} = -2\frac{\delta S}{\delta\bar{g}^{ab}}\bar{\nabla}^a\xi^b \tag{B.6}$$

$$\begin{aligned}\bar{\nabla}_i\left(\frac{\partial\mathscr{L}}{\partial\partial_i\bar{g}^{ab}}\delta\bar{g}^{ab}\right) =& -2\bar{\nabla}_i\left(\frac{\partial\mathscr{L}}{\partial\partial_i\bar{g}^{ab}}\bar{\nabla}^a\xi^b\right)\\ =& -2\bar{\nabla}_i\left(\frac{\partial\mathscr{L}}{\partial\partial_i\bar{g}^{ab}}\right)\bar{\nabla}^a\xi^b - 2\left(\frac{\partial\mathscr{L}}{\partial\partial_i\bar{g}^{ab}}\right)\bar{\nabla}_i\bar{\nabla}^a\xi^b\end{aligned} \tag{B.7}$$

For concreteness, we suppose that φ is a scalar field, such that

$$\delta\varphi = \xi^c\partial_c\varphi = \xi^c\bar{\nabla}_c\varphi\ , \tag{B.8}$$

and obtain the remaining terms in (B.2):

$$\frac{\delta S}{\delta\varphi}\delta\varphi = \frac{\delta S}{\delta\varphi}\xi^c\bar{\nabla}_c\varphi \tag{B.9}$$

$$\frac{\partial\mathscr{L}}{\partial\bar{\nabla}_i\varphi}\delta\varphi = \frac{\partial\mathscr{L}}{\partial\bar{\nabla}_i\varphi}\xi^c\bar{\nabla}_c\varphi \tag{B.10}$$

Writing (3.22) in terms of the covariant derivative,

$$\delta\mathscr{L} = \partial_c\left(\xi^c\mathscr{L}\right) = \bar{\nabla}_c\left(\xi^c\mathscr{L}\right) = \xi^c\bar{\nabla}_c\mathscr{L} + \mathscr{L}\bar{\nabla}_c\xi^c\ , \tag{B.11}$$

and equating with (B.2) after substituting equations (B.4) - (B.10), we have

$$\begin{aligned}0 =& \left[\frac{\delta S}{\delta h^{ab}}\bar{\nabla}_c h^{ab} + \frac{\delta S}{\delta\varphi}\bar{\nabla}_c\varphi - \bar{\nabla}_i\left(\delta^i_c\mathscr{L} - \frac{\partial\mathscr{L}}{\partial\bar{\nabla}_i h^{ab}}\bar{\nabla}_c h^{ab} - \frac{\partial\mathscr{L}}{\partial\bar{\nabla}_i\varphi}\bar{\nabla}_c\varphi\right)\right]\xi^c\\ &- \left[2\frac{\delta S}{\delta\bar{g}^{ab}}\bar{g}^{ac} + \left(\delta^c_b\mathscr{L} - \frac{\partial\mathscr{L}}{\partial\bar{\nabla}_c h^{ad}}\bar{\nabla}_b h^{ad} - \frac{\partial\mathscr{L}}{\partial\bar{\nabla}_c\varphi}\bar{\nabla}_b\varphi\right) + 2\frac{\delta S}{\delta h^{ab}}h^{ac}\right.\\ &\left.+ 2\bar{\nabla}_i\left(\frac{\partial\mathscr{L}}{\partial\partial_i\bar{g}^{ab}}\bar{g}^{ac} + \frac{\partial\mathscr{L}}{\partial\bar{\nabla}_i h^{ab}}h^{ca}\right)\right]\bar{\nabla}_c\xi^b - 2\left[\frac{\partial\mathscr{L}}{\partial\partial_i\bar{g}^{ab}}\bar{g}^{ac} + \frac{\partial\mathscr{L}}{\partial\bar{\nabla}_i h^{ab}}h^{ca}\right]\bar{\nabla}_i\bar{\nabla}_c\xi^b\end{aligned} \tag{B.12}$$

In the first line of the equation above, all terms cancel out since it consists of the r.h.s. of equality (3.18) being subtracted to the l.h.s.. The last term also vanishes identically since $\bar{\nabla}_i\bar{\nabla}_c\xi^b$ is symmetric in the indices i and c whilst the coefficient in front of it ($\equiv \sqrt{-|\bar{g}|}\psi_b{}^{ci}$) is antisymmetric ($\psi_b{}^{ci} = -\psi_b{}^{ic}$). All that remains is the term proportional to $\bar{\nabla}_c\xi^b$ and we see that its coefficient must vanish due to arbitrariness of the transformation parameter:

$$-\sqrt{-|\bar{g}|}T^{ca}_{\mathrm{Ros}}\bar{g}_{ab} + \sqrt{-|\bar{g}|}T^{ca}_{\mathrm{Can}}\bar{g}_{ab} - 2\frac{\delta S}{\delta h^{ab}}h^{ac} + \sqrt{-|\bar{g}|}\bar{\nabla}_a\psi_b{}^{ca} = 0 \tag{3.36}$$

where we used definitions (3.29) and (3.19).

C Solving for non-minimal coupling term

Dividing (4.60) by $\sqrt{-|\gamma|}$ we get

$$\tilde{\Delta}^c{}_{pq\mu}{}^{\upsilon}{}_{(a}K_{b)\upsilon}{}^{\mu}{}_{ef}{}^{d} + \frac{\partial K_{pq}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{ab}} - \frac{1}{2}K_{pq}{}^{c}{}_{ef}{}^{d}\gamma_{ab} = -\tilde{\Delta}^c{}_{pq\tau}{}^{i}{}_{j}Q_i{}^{\tau[dj]}{}_{abef} \tag{C.1}$$

Substituting the following equations in the equation above, one sees that it's identically satisfied.

$$\begin{aligned}\frac{\partial K_{pq}{}^{c}{}_{ef}{}^{d}}{\partial\gamma^{ab}} = \frac{1}{2}\Big[&\delta^c_{(a}\delta^d_{b)}\gamma_{p(e}\gamma_{f)q} - \gamma^{cd}\gamma_{p(a}\gamma_{b)(e}\gamma_{f)q} - \gamma^{cd}\gamma_{p(e}\gamma_{f)(a}\gamma_{b)q} - \delta^c_{(a}\delta^d_{b)}\gamma_{pq}\gamma_{ef}\\ &+ \gamma^{cd}\gamma_{p(a}\gamma_{b)q}\gamma_{ef} + \gamma^{cd}\gamma_{e(a}\gamma_{b)f}\gamma_{pq} + 2\delta^c_{(e}\gamma_{f)(a}\gamma_{b)(p}\delta^d_{q)} - \delta^c_{(e}\delta^d_{f)}\gamma_{p(a}\gamma_{b)q} - \delta^c_{(p}\delta^d_{q)}\gamma_{e(a}\gamma_{b)f}\Big]\end{aligned} \tag{C.2}$$

$$\begin{aligned}\tilde{\Delta}^c{}_{pq\mu}{}^{\alpha}{}_{a}K_{b\alpha}{}^{\mu}{}_{ef}{}^{d} = \frac{1}{4}\Big(&2\delta^c_{(e}\delta^d_{f)}\gamma_{a(p}\gamma_{q)b} - 4\delta^c_{(e}\gamma_{f)b}\gamma_{a(p}\delta^d_{q)} + 3\delta^c_b\gamma_{a(p}\delta^d_{q)}\gamma_{ef} - \delta^c_a\delta^d_{(p}\gamma_{q)b}\gamma_{ef}\\ &+ 4\delta^c_{[a}\gamma_{b](p}\gamma_{q)(e}\delta^d_{f)} + \delta^c_a\delta^d_b\left[\gamma_{ef}\gamma_{pq} - \gamma_{ep}\gamma_{qf} - \gamma_{eq}\gamma_{fp}\right] - 3\gamma^{cd}\gamma_{ef}\gamma_{a(p}\gamma_{q)b} + 4\gamma^{cd}\gamma_{b(e}\gamma_{f)(q}\gamma_{p)a}\Big)\end{aligned} \tag{C.3}$$

$$\begin{aligned}-\tilde{\Delta}^c{}_{pq\beta}{}^{i}{}_{j}Q_i{}^{\beta[dj]}{}_{abef} = \frac{1}{4}\Big(&\left[\gamma_{ab}\gamma_{ef} + \gamma_{ae}\gamma_{fb} + \gamma_{af}\gamma_{be}\right]\left[\gamma^{cd}\gamma_{pq} - \delta^c_{(p}\delta^d_{q)}\right]\\ &+\gamma_{ab}\left[2\delta^c_{(e}\gamma_{f)(p}\delta^d_{q)} - \gamma_{pq}\delta^c_{(e}\delta^d_{f)} - \gamma^{cd}\gamma_{e(p}\gamma_{q)f}\right] + (ab)\leftrightarrow(ef)\Big)\end{aligned} \tag{C.4}$$

Q in this last equation is given by (4.62).

D Extra iteration: “metric self-coupling” of the mass term

$$\begin{aligned}\frac{\delta\tilde{S}_2^{\rm m}}{\delta\gamma^{ab}} = -\frac{m^2}{48}\sqrt{-|\gamma|}\Bigg[\frac{1}{2}\gamma_{ab}\Big(&h^4 + 12h_{st}h^{st}h^2 + 16k'h_i^e h_e^c h_c^i h + 24k'h^{st}h_{se}h^{ed}h_{dt}\\ &+6k'h_{st}h^{st}h_{ef}h^{ef}\Big) + 4\Big(h_{ab}h^3 + 6h_a^d h_{bd}h^2 + 6h_{st}h^{st}h_{ab}h + 12k'h_{ad}h^{df}h_{fb}h\\ &+4k'h_e^s h_s^c h_c^e h_{ab} + 24k'h_{aj}h_b^d h_d^f h_f^j + 6k'h_a^d h_{bd}h_{st}h^{st}\Big)\Bigg]\end{aligned} \tag{D.1}$$

From this we get five consistency requirements that are identically satisfied (two of them are equivalent to (5.71)).

E Proof: partial actions below quadratic order vanish

$$\begin{aligned}S[\bar{e};f] &= \sum_{n=0}^{\infty}\frac{\chi^n}{n!}\int d^Dx_n ... \, d^Dx_1 \frac{\delta^n S[e]}{\delta e^c{}_C(x_n)...\,\delta e^b{}_B(x_1)}\bigg|_{e=\bar{e}} f^b{}_B(x_1)...\, f^c{}_C(x_n)\\ &\equiv \sum_{n=0}^{\infty}\chi^n\int d^Dx_n\,\mathscr{L}^{(n)}\end{aligned} \tag{E.1}$$

(6.32) implies that

$$\mathscr{L}^{(0)} = K^b{}_a{}^{Ac}{}_d{}^D[\bar{e}]\bar{\nabla}_b\bar{e}^a{}_A\bar{\nabla}_c\bar{e}^d{}_D \tag{E.2}$$

and taking into account invariance under (infinitesimal) LLT[28], we have

$$\begin{aligned}0 = \hat{\delta}\mathscr{L}^{(0)} &= 2K^b{}_a{}^{Ac}{}_d{}^D[\bar{e}]\bar{\nabla}_b\hat{\delta}\bar{e}^a{}_A\bar{\nabla}_c\bar{e}^d{}_D + \frac{\partial K^b{}_a{}^{Ac}{}_d{}^D[\bar{e}]}{\partial\bar{e}^i{}_I}\hat{\delta}\bar{e}^i{}_I\bar{\nabla}_b\bar{e}^a{}_A\bar{\nabla}_c\bar{e}^d{}_D\\ &= \left[\left(2K^b{}_a{}^{Ic}{}_d{}^D[\bar{e}]\bar{\nabla}_b\bar{e}^a{}_B + \frac{\partial K^b{}_a{}^{Ac}{}_d{}^D[\bar{e}]}{\partial\bar{e}^i{}_I}\bar{e}^i{}_B\bar{\nabla}_b\bar{e}^a{}_A\right)\epsilon^B{}_I + 2\left(K^b{}_a{}^{Ac}{}_d{}^D[\bar{e}]\bar{e}^{aB}\right)\bar{\nabla}_b\epsilon_{BA}\right]\bar{\nabla}_c\bar{e}^d{}_D\end{aligned} \tag{E.3}$$

where we used $\hat{\delta}\bar{e}^a{}_A = \bar{e}^a{}_B\epsilon^B{}_A$. This implies, using arbitrariness of the transformation parameter and its antisymmetry,

$$K^b{}_a{}^{[Ac}{}_d{}^D[\bar{e}]\bar{e}^{aB]} = 0 \tag{E.4}$$

This leads to (using $\bar{\omega}_{bAB}$'s antisymmetry)

$$\mathscr{L}^{(0)} = K^b{}_a{}^{Ac}{}_d{}^D[\bar{e}]\bar{\omega}_{bAB}\bar{e}^{aB}\bar{\nabla}_c\bar{e}^d{}_D = K^b{}_a{}^{[Ac}{}_d{}^D[\bar{e}]\bar{e}^{aB]}\bar{\nabla}_c\bar{e}^d{}_D\bar{\omega}_{bAB} = 0 \tag{E.5}$$

[28] Γ is a Lorentz scalar.

Moving on to $n = 1$:

$$\int d^D x_1 \mathscr{L}^{(1)} = \int d^D x_1 \frac{\delta^n S[e]}{\delta e^b{}_B(x_1)}\bigg|_{e=\bar{e}} f^b{}_B(x_1)$$
$$= \int d^D x_1 \left[2 \int d^D x K^b{}_a{}^{Ac}{}_d{}^D[\bar{e}]\bar{\nabla}_b \bar{e}^a{}_A \frac{\delta \nabla_c e^d{}_D}{\delta e^b{}_B(x_1)}\bigg|_{e=\bar{e}} f^b{}_B(x_1) + \frac{\partial K^b{}_a{}^{Ac}{}_d{}^D}{\partial e^i{}_I}\bigg|_{e=\bar{e}} f^i{}_I \bar{\nabla}_b \bar{e}^a{}_A \bar{\nabla}_c \bar{e}^d{}_D\right] \tag{E.6}$$

The first term vanishes when we use (E.4) as in (E.5) and we have

$$\mathscr{L}^{(1)} = \frac{\partial K^b{}_a{}^{Ac}{}_d{}^D[\bar{e}]}{\partial \bar{e}^i{}_I} f^i{}_I \bar{\nabla}_b \bar{e}^a{}_A \bar{\nabla}_c \bar{e}^d{}_D \equiv K'^b{}_a{}^{Ac}{}_d{}^D{}_i{}^I[\bar{e}] f^i{}_I \bar{\nabla}_b \bar{e}^a{}_A \bar{\nabla}_c \bar{e}^d{}_D \tag{E.7}$$

Due to Lorentz invariance,

$$0 = \hat{\delta}\mathscr{L}^{(1)} = 2K'^b{}_a{}^{Ac}{}_d{}^D{}_i{}^I[\bar{e}] f^i{}_I \bar{\nabla}_b \hat{\delta}\bar{e}^a{}_A \bar{\nabla}_c \bar{e}^d{}_D + K'^b{}_a{}^{Ac}{}_d{}^D{}_i{}^I[\bar{e}]\hat{\delta} f^i{}_I \bar{\nabla}_b \bar{e}^a{}_A \bar{\nabla}_c \bar{e}^d{}_D$$
$$+ \frac{\partial K'^b{}_a{}^{Ac}{}_d{}^D{}_i{}^I[\bar{e}]}{\partial \bar{e}^u{}_U}\hat{\delta}\bar{e}^u{}_U f^i{}_I \bar{\nabla}_b \bar{e}^a{}_A \bar{\nabla}_c \bar{e}^d{}_D = \epsilon^B{}_U(\ldots) + \bar{\nabla}_b \epsilon_{BA}\left(2K'^b{}_a{}^{Ac}{}_d{}^D{}_i{}^I[\bar{e}] f^i{}_I \bar{e}^{aB}\right)\bar{\nabla}_c \bar{e}^d{}_D \tag{E.8}$$

similarly to (E.3). Using arbitrariness of the transformation parameter and its antisymmetry once more, this implies

$$K'^b{}_a{}^{[Ac}{}_d{}^D{}_i{}^I[\bar{e}] f^i{}_I \bar{e}^{aB]} = 0 \tag{E.9}$$

One easily sees from (E.5) that $\mathscr{L}^{(1)}$ also vanishes.

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